11
Arithmetic Progression

ARITHMETIC PROGRESSION

SEQUENCE *A succession of numbers arranged in a definite order according to a certain given rule is called a sequence.*

The number occurring at the nth place of a sequence is called its *nth term* or the *general term*, to be denoted by a_n.

A sequence is said to be finite or infinite according as the number of terms in it is finite or infinite respectively.

By adding the terms of a sequence, we get a series.

A series is said to be finite or infinite according as the number of terms in it is finite or infinite respectively.

EXAMPLE 1 *Write first five terms of the sequence given by the rule* $a_n = (2n+1)$ *and obtain the corresponding series.*

SOLUTION We have, $a_n = (2n+1)$. ... (i)

Putting $n = 1, 2, 3, 4, 5, \ldots$ successively in (i), we get

$$a_1 = (2\times 1+1) = 3;\ a_2 = (2\times 2+1) = 5;\ a_3 = (2\times 3+1) = 7;$$
$$a_4 = (2\times 4+1) = 9 \quad \text{and} \quad a_5 = (2\times 5+1) = 11.$$

Hence, the required sequence is 3, 5, 7, 9, 11, ...

The corresponding series is $3+5+7+9+11+\ldots$.

EXAMPLE 2 *Write first four terms of the sequence given by* $a_n = \frac{1}{6}(2n-3)$ *and obtain the corresponding series.*

SOLUTION We have, $a_n = \frac{1}{6}(2n-3)$. ... (i)

Putting $n = 1, 2, 3, 4, \ldots$ successively in (i), we get

$$a_1 = \frac{1}{6}(2\times 1-3) = \frac{-1}{6};\ a_2 = \frac{1}{6}(2\times 2-3) = \frac{1}{6};$$
$$a_3 = \frac{1}{6}(2\times 3-3) = \frac{3}{6} = \frac{1}{2};\ a_4 = \frac{1}{6}(2\times 4-3) = \frac{5}{6}.$$

Hence, the required sequence is $\frac{-1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \ldots$.

The corresponding series is $-\frac{1}{6}+\frac{1}{6}+\frac{1}{2}+\frac{5}{6}+\ldots$.

EXAMPLE 3 *The Fibonacci sequence is defined by*

$$1 = a_1 = a_2 \text{ and } a_n = a_{n-1} + a_{n-2},\ n > 2.$$

Find $\frac{a_{n+1}}{a_n}$ *for* $n = 1, 2, 3, 4, 5.$

SOLUTION We have, $a_1 = 1, a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n > 2.$

$\therefore \quad a_3 = (a_2 + a_1) = (1 + 1) = 2, a_4 = (a_3 + a_2) = (2 + 1) = 3,$

$a_5 = (a_4 + a_3) = (3 + 2) = 5.$

$\therefore \quad \frac{a_2}{a_1} = 1, \frac{a_3}{a_2} = \frac{2}{1} = 2, \frac{a_4}{a_3} = \frac{3}{2}$ and $\frac{a_5}{a_4} = \frac{5}{3}.$

PROGRESSIONS

Sequences following certain patterns are called progressions.

ARITHMETIC PROGRESSION (AP)

It is a sequence in which each term except the first one differs from its preceding term by a constant.

This constant difference is called the common difference of the AP.

In an AP we usually denote the *first term* by *a*, the *common difference* by *d* and the *nth term* by T_n.

SOME EXAMPLES OF AP

EXAMPLE 1 *Show that the sequence defined by* $T_n = 3n + 5$ *is an AP. Find its common difference.*

SOLUTION We have, $T_n = 3n + 5.$ … (i)

Replacing *n* by $(n - 1)$ in (i), we get

$$T_{n-1} = 3(n - 1) + 5 \Rightarrow T_{n-1} = 3n + 2. \quad \text{… (ii)}$$

Subtracting (ii) from (i), we get

$(T_n - T_{n-1}) = (3n + 5) - (3n + 2) = 3$, which is constant.

Hence, the given sequence is an AP with common difference 3.

EXAMPLE 2 *Show that the sequence* $\log a, \log\left(\frac{a^2}{b}\right), \log\left(\frac{a^3}{b^2}\right), \log\left(\frac{a^4}{b^3}\right), \ldots$ *forms an AP. Find its common difference.*

SOLUTION By symmetry, we find that

$$T_n = \log\left(\frac{a^n}{b^{n-1}}\right) \text{ and } T_{n-1} = \log\left(\frac{a^{n-1}}{b^{n-2}}\right).$$

$$\therefore \quad (T_n - T_{n-1}) = \log\left(\frac{a^n}{b^{n-1}}\right) - \log\left(\frac{a^{n-1}}{b^{n-2}}\right)$$

$$= \log\left(\frac{a^n}{b^{n-1}} \times \frac{b^{n-2}}{a^{n-1}}\right) = \log\left(\frac{a}{b}\right) = \text{constant}.$$

Hence, the given sequence in an AP with common difference $\log\left(\frac{a}{b}\right).$

EXAMPLE 3 *Show that the sequence defined by* $T_n = 3n^2 + 2$ *is not an AP.*

SOLUTION We have

$$T_n = 3n^2 + 2 \text{ and } T_{n-1} = 3(n-1)^2 + 2 \Rightarrow T_{n-1} = 3n^2 - 6n + 5.$$

$$\therefore \quad (T_n - T_{n-1}) = (3n^2 + 2) - (3n^2 - 6n + 5) \Rightarrow (T_n - T_{n-1}) = (6n - 3).$$

This shows that $(T_n - T_{n-1})$ is not independent of *n* and therefore, it is not constant.

Hence, the given sequence is not an AP.

GENERAL TERM OF AN AP

THEOREM 1 *Show that the nth term of an AP with first term a and common difference d is given by* $T_n = a + (n-1)d$.

PROOF Let us consider an AP with first term *a* and common difference *d*.

Then, the given AP is

$$a, (a+d), (a+2d), (a+3d), (a+4d), \ldots .$$

In this AP, we have

first term, $T_1 = a = (a + 0 \times d) = a + (1-1)d$;

second term, $T_2 = (a+d) = a + (2-1)d$;

third term, $T_3 = (a+2d) = a + (3-1)d$;

....

....

*n*th term, $T_n = a + (n-1)d$.

Hence, $T_n = a + (n-1)d$.

This is called the general term of the AP.

> In an AP with first term = *a* and common difference = *d*, we have
> $$T_n = a + (n-1)d.$$

SOME SIMPLE PROPERTIES OF AN AP

(i) If a constant is added to each term of an AP then the resulting progression is an AP.

(ii) If a constant is subtracted from each term of an AP then the resulting progression is an AP.

(iii) If each term of an AP is multiplied by the same nonzero number then the resulting progression is an AP.

(iv) If each term of an AP is divided by the same nonzero number then the resulting progression is an AP.

SOLVED EXAMPLES

EXAMPLE 1 *Show that the progression* 7, 12, 17, 22, 27, ... *is an AP. Find its general term and the* 14*th term.*

SOLUTION We have

$(12-7) = (17-12) = (22-17) = (27-22) = 5$, which is constant.

So, the given progression is an AP in which $a = 7$ and $d = 5$.

Its general term, $T_n = \{a + (n-1)d\} = 7 + (n-1) \times 5 \Rightarrow T_n = (5n+2)$.

$\therefore$ 14th term, $T_{14} = (5 \times 14 + 2) = 72$.

EXAMPLE 2 *Show that the progression* $\log a, \log (ab), \log (ab^2), \log (ab^3), \ldots$ *is an AP. Find its general term and the* 10*th term.*

SOLUTION We have

$$\log (ab) - \log a = \log a + \log b - \log a = \log b,$$

$$\log (ab^2) - \log (ab) = (\log a + 2\log b) - (\log a + \log b) = \log b,$$

$$\log (ab^3) - \log (ab^2) = (\log a + 3\log b) - (\log a + 2\log b) = \log b.$$

$\therefore$ $(T_2 - T_1) = (T_3 - T_2) = (T_4 - T_3) = \log b =$ constant.

So, the given progression is an AP.

Let *A* be the first term and *d* be the common difference of this AP.

Then, $A = \log a$ and $d = \log b$.

$\therefore$ general term, $T_n = A + (n-1)d$

$\Rightarrow$ $T_n = \log a + (n-1)\log b$.

And, 10th term, $T_{10} = \log a + (10-1)\log b = \log a + 9\log b$

$\Rightarrow$ $T_{10} = \log a + \log (b^9) \Rightarrow T_{10} = \log (ab^9)$.

EXAMPLE 3 *Show that the progression* 21, 16, 11, 6, 1, … *is an AP. Write its first term and the common difference. Which term of this AP is* –54*?*

SOLUTION The given progression is 21, 16, 11, 6, 1, … .

We have

$(16-21) = (11-16) = (6-11) = (1-6) = -5$, which is constant.

So, the given progression is an AP.

Its first term, $a = 21$ and common difference, $d = -5$.

Let its *n*th term be –54. Then,

$$T_n = -54 \Rightarrow a + (n-1)d = -54$$

$$\Rightarrow 21 + (n-1) \times (-5) = -54$$

$$\Rightarrow -5n + 26 = -54 \Rightarrow 5n = 80 \Rightarrow n = 16.$$

$\therefore$ $T_{16} = -54$.

Hence, the 16th term of the given AP is –54.

EXAMPLE 4 *Show that the progression* –11, –7, –3, 1, 5, …, 161 *is an AP. How many terms does it have?*

SOLUTION We have

$(-7) - (-11) = (-7 + 11) = 4$, $(-3) - (-7) = (-3 + 7) = 4$,

$1 - (-3) = (1 + 3) = 4$ and $(5 - 1) = 4$.

$\therefore$ $\{(-7) - (-11)\} = \{(-3) - (-7)\} = \{1 - (-3)\} = (5 - 1) = 4$, which is constant.

So, the given progression is an AP in which $a = -11$ and $d = 4$.

Let it have n terms. Then,

$$T_n = 161 \Rightarrow a + (n-1)d = 161$$
$$\Rightarrow (-11) + (n-1) \times 4 = 161$$
$$\Rightarrow 4n = 176 \Rightarrow n = 44.$$

$\therefore$ $T_{44} = 161$ and hence there are 44 terms in the given AP.

EXAMPLE 5 *Is* 319 *a term of the AP* 11, 17, 23, 29, 35, ...?

SOLUTION In the given AP, we have $a = 11$ and $d = (17 - 11) = 6$.

If possible, let the nth term of this AP be 319. Then,

$$T_n = 319 \Rightarrow a + (n-1)d = 319$$
$$\Rightarrow 11 + (n-1) \times 6 = 319$$
$$\Rightarrow 6n + 5 = 319 \Rightarrow 6n = 314$$
$$\Rightarrow 3n = 157 \Rightarrow n = \frac{157}{3} = 52\frac{1}{3}.$$

Since the number of terms cannot be a fraction, so it follows that 319 is not a term of the given AP.

EXAMPLE 6 *Which term of the AP* $30, 29\frac{1}{4}, 28\frac{1}{2}, 27\frac{3}{4}, \ldots$ *is the first negative term?*

SOLUTION We have

$$\left(\frac{117}{4} - 30\right) = \left(\frac{57}{2} - \frac{117}{4}\right) = \left(\frac{111}{4} - \frac{57}{2}\right) = \frac{-3}{4}.$$

$\therefore$ $a = 30$ and $d = \frac{-3}{4}$.

Let its nth term be the first negative term. Then,

$$T_n < 0 \Rightarrow a + (n-1)d < 0$$
$$\Rightarrow 30 + (n-1) \times \left(\frac{-3}{4}\right) < 0$$
$$\Rightarrow 123 - 3n < 0 \Rightarrow 123 < 3n$$
$$\Rightarrow 3n > 123 \Rightarrow n > 41.$$

Hence, the 42nd term of the given AP is the first negative term.

And, $T_{42} = a + (42-1)d = 30 + 41 \times \left(\frac{-3}{4}\right) = \frac{-3}{4}$.

Hence, the first negative term of the given AP is $\frac{-3}{4}$.

EXAMPLE 7 *Find the* 10*th common term between the arithmetic series* $3 + 7 + 11 + 15 + \ldots$ *and* $1 + 6 + 11 + 16 + \ldots$.

SOLUTION Clearly, the first common term in the two series is 11.

LCM of the common differences of the two series = LCM (4, 5) = 20.

Let us now consider an AP in which $a = 11$ and $d = 20$.

Every term of this AP is a common term of the two given series.

So, the required 10th common term

$$= \{11 + (10 - 1) \times 20\} = (11 + 180) = 191.$$

Hence, 191 is the 10th common term of the two given series.

EXAMPLE 8 *If $a_1, a_2, a_3, \ldots, a_n$ are in AP with common difference d (where $d \neq 0$) then prove that the sum of the series $\sin d(\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \ldots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n)$ is equal to $(\cot a_1 - \cot a_n)$.*

SOLUTION Since $a_1, a_2, a_3, \ldots, a_n$ are in AP with common difference d, we have

$$(a_2 - a_1) = (a_3 - a_2) = (a_4 - a_3) = \ldots = (a_n - a_{n-1}) = d.$$

$$\therefore \quad \sin d(\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \ldots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n)$$

$$= \frac{\sin d}{\sin a_1 \sin a_2} + \frac{\sin d}{\sin a_2 \sin a_3} + \frac{\sin d}{\sin a_3 \sin a_4} + \ldots + \frac{\sin d}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin (a_2 - a_1)}{\sin a_1 \sin a_2} + \frac{\sin (a_3 - a_2)}{\sin a_2 \sin a_3} + \frac{\sin (a_4 - a_3)}{\sin a_3 \sin a_4} + \ldots + \frac{\sin (a_n - a_{n-1})}{\sin a_{n-1} \sin a_n}$$

$$[\because \ d = (a_2 - a_1) = (a_3 - a_2) = \ldots = (a_n - a_{n-1})]$$

$$= \frac{\sin a_2 \cos a_1 - \cos a_2 \sin a_1}{\sin a_1 \sin a_2} + \frac{\sin a_3 \cos a_2 - \cos a_3 \sin a_2}{\sin a_2 \sin a_3} + \ldots + \frac{\sin a_n \cos a_{n-1} - \cos a_n \sin a_{n-1}}{\sin a_{n-1} \sin a_n}$$

$$= \left(\frac{\cos a_1}{\sin a_1} - \frac{\cos a_2}{\sin a_2}\right) + \left(\frac{\cos a_2}{\sin a_2} - \frac{\cos a_3}{\sin a_3}\right) + \ldots + \left(\frac{\cos a_{n-1}}{\sin a_{n-1}} - \frac{\cos a_n}{\sin a_n}\right)$$

$$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + (\cot a_3 - \cot a_4) + \ldots + (\cot a_{n-1} - \cot a_n)$$

$$= (\cot a_1 - \cot a_n).$$

Hence, $\sin d(\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \ldots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n) = (\cot a_1 - \cot a_n)$.

EXAMPLE 9 *If $a_1, a_2, a_3, \ldots, a_n$ are in AP, where $a_i > 0$ for each i, show that*

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{(n-1)}{\sqrt{a_1} + \sqrt{a_n}}.$$

SOLUTION Let d be the common difference of the given AP. Then,

$$(a_2 - a_1) = (a_3 - a_2) = \ldots = (a_n - a_{n-1}) = d.$$

$$\text{Now, LHS} = \frac{1}{(\sqrt{a_2} + \sqrt{a_1})} + \frac{1}{(\sqrt{a_3} + \sqrt{a_2})} + \ldots + \frac{1}{(\sqrt{a_n} + \sqrt{a_{n-1}})}$$

$$= \frac{(\sqrt{a_2} - \sqrt{a_1})}{(\sqrt{a_2} + \sqrt{a_1})(\sqrt{a_2} - \sqrt{a_1})} + \frac{(\sqrt{a_3} - \sqrt{a_2})}{(\sqrt{a_3} + \sqrt{a_2})(\sqrt{a_3} - \sqrt{a_2})} + \ldots + \frac{(\sqrt{a_n} - \sqrt{a_{n-1}})}{(\sqrt{a_n} + \sqrt{a_{n-1}})(\sqrt{a_n} - \sqrt{a_{n-1}})}$$

$$= \frac{(\sqrt{a_2}-\sqrt{a_1})}{(a_2-a_1)} + \frac{(\sqrt{a_3}-\sqrt{a_2})}{(a_3-a_2)} + \ldots + \frac{(\sqrt{a_n}-\sqrt{a_{n-1}})}{(a_n-a_{n-1})}$$

$$= \frac{(\sqrt{a_2}-\sqrt{a_1})}{d} + \frac{(\sqrt{a_3}-\sqrt{a_2})}{d} + \ldots + \frac{(\sqrt{a_n}-\sqrt{a_{n-1}})}{d}$$

$$[\because\ (a_2-a_1) = (a_3-a_2) = (a_4-a_3) = \ldots = (a_n-a_{n-1}) = d]$$

$$= \frac{1}{d}\cdot\{(\sqrt{a_2}-\sqrt{a_1}) + (\sqrt{a_3}-\sqrt{a_2}) + (\sqrt{a_4}-\sqrt{a_3}) + \ldots + (\sqrt{a_n}-\sqrt{a_{n-1}})\}$$

$$= \frac{1}{d}\cdot(\sqrt{a_n}-\sqrt{a_1}) = \frac{1}{d}\cdot\left\{(\sqrt{a_n}-\sqrt{a_1})\times\frac{(\sqrt{a_n}+\sqrt{a_1})}{(\sqrt{a_n}+\sqrt{a_1})}\right\}$$

$$= \frac{1}{d}\cdot\left\{\frac{(a_n-a_1)}{(\sqrt{a_n}+\sqrt{a_1})}\right\} = \frac{1}{d}\cdot\left\{\frac{a_1+(n-1)d-a_1}{\sqrt{a_1}+\sqrt{a_n}}\right\}$$

$$[\because\ a_n = a_1+(n-1)d]$$

$$= \frac{1}{d}\cdot\left\{\frac{(n-1)d}{\sqrt{a_1}+\sqrt{a_n}}\right\} = \frac{(n-1)}{(\sqrt{a_1}+\sqrt{a_n})} = \text{RHS}.$$

Thus, LHS = RHS and hence the result follows.

EXAMPLE 10 *If $a_1, a_2, a_3, \ldots, a_n$ be an AP of nonzero terms then prove that*

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \frac{1}{a_3a_4} + \ldots + \frac{1}{a_{n-1}a_n} = \frac{(n-1)}{a_1a_n}.$$

SOLUTION Let d be the common difference of the given AP. Then,

$$(a_2-a_1) = (a_3-a_2) = (a_4-a_3) = \ldots = (a_n-a_{n-1}) = d.$$

$$\therefore\quad \frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \frac{1}{a_3a_4} + \ldots + \frac{1}{a_{n-1}a_n}$$

$$= \frac{1}{d}\cdot\left\{\frac{d}{a_1a_2} + \frac{d}{a_2a_3} + \frac{d}{a_3a_4} + \ldots + \frac{d}{a_{n-1}a_n}\right\}$$

$$= \frac{1}{d}\cdot\left\{\frac{(a_2-a_1)}{a_1a_2} + \frac{(a_3-a_2)}{a_2a_3} + \frac{(a_4-a_3)}{a_3a_4} + \ldots + \frac{(a_n-a_{n-1})}{a_{n-1}a_n}\right\}$$

$$[\because\ (a_2-a_1) = (a_3-a_2) = \ldots = (a_n-a_{n-1}) = d]$$

$$= \frac{1}{d}\cdot\left\{\left(\frac{1}{a_1}-\frac{1}{a_2}\right) + \left(\frac{1}{a_2}-\frac{1}{a_3}\right) + \left(\frac{1}{a_3}-\frac{1}{a_4}\right) + \ldots + \left(\frac{1}{a_{n-1}}-\frac{1}{a_n}\right)\right\}$$

$$= \frac{1}{d}\cdot\left(\frac{1}{a_1}-\frac{1}{a_n}\right) = \frac{1}{d}\cdot\frac{(a_n-a_1)}{a_1a_n}$$

$$= \frac{1}{d}\cdot\left[\frac{\{a_1+(n-1)d\}-a_1}{a_1a_n}\right] \qquad [\because\ a_n = a_1+(n-1)d]$$

$$= \frac{1}{d}\cdot\frac{(n-1)d}{a_1a_n} = \frac{(n-1)}{a_1a_n}.$$

Hence, the result follows.

EXAMPLE 11 *In an AP, the pth term is q and $(p+q)$th term is 0. Then, prove that its qth term is p.*

SOLUTION Let a be the first term and d be the common difference of the given AP. Then,

$$T_p = q \Rightarrow a + (p-1)d = q \quad \ldots \text{(i)}$$

and $$T_{p+q} = 0 \Rightarrow a + (p+q-1)d = 0. \quad \ldots \text{(ii)}$$

On subtracting (i) from (ii), we get

$$qd = -q \Rightarrow d = -1.$$

Putting $d = -1$ in (i), we get $a = (q + p - 1)$.

$$\therefore \quad T_q = a + (q-1)d = (q+p-1) + (q-1)\times(-1) \Rightarrow T_q = p$$

$$[\because\ a = (q+p-1) \text{ and } d = -1].$$

Hence, the qth term is p.

TO FIND THE nTH TERM FROM THE END OF AN AP

THEOREM 2 *Show that the nth term from the end of an AP with the first term a, common difference d and the last term l is given by $\{l - (n-1)d\}$.*

PROOF Let a be the first term, d be the common difference and l be the last term of a given AP.

Then, the AP is given by

$$a, a+d, a+2d, \ldots, (l-2d), (l-d), l.$$

$\therefore$ last term $= l = l - (1-1)d$

2nd term from the end $= (l-d) = l - (2-1)d$

3rd term from the end $(l-2d) = l - (3-1)d$

… … … … … … … … …

… … … … … … … … …

$\therefore$ nth term from the end $= l - (n-1)d$.

Hence, nth term from the end $= l - (n-1)d$.

EXAMPLE 12 *Find the 17th term from the end of the AP $-36, -31, -26, -21, \ldots, 79$.*

SOLUTION Here $a = 36$, $d = \{-31 - (-36)\} = 5$ and $l = 79$.

$$\therefore \quad \text{17th term from the end} = l - (n-1)d$$

$$= 79 - (17-1)\times 5 = (79 - 80) = -1.$$

Hence, the 17th term from the end is –1.

SOME RESULTS ON AP

THEOREM 1 *If the nth term of a progression is a linear expression in n then show that it is an AP.*

PROOF Let the nth term of a progression be a linear expression in n.

Then, $T_n = an + b$, … (i)

where a and b are constants.

Replacing n by $(n-1)$ in (i), we get

$$T_{n-1} = a(n-1) + b. \quad \ldots \text{(ii)}$$

On subtracting (ii) from (i), we get

$(T_n - T_{n-1}) = a$, which is constant.

This shows that the difference between any two consecutive terms of the given progression is constant.

Hence, the given progression is an AP.

THEOREM 2 *If the mth term of an AP be* $(1/n)$ *and its nth term be* $(1/m)$ *then show that its (mn)th term is* 1.

PROOF In the given AP, let the first term = a and the common difference = d.

Then, $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$

$\Rightarrow \quad a + (m-1)d = \frac{1}{n} \quad \ldots$ (i) and $a + (n-1)d = \frac{1}{m}$. $\ldots$ (ii)

On subtracting (ii) from (i), we get

$$(m-n)d = \left(\frac{1}{n} - \frac{1}{m}\right) \Rightarrow (m-n)d = \frac{(m-n)}{mn}$$

$$\Rightarrow d = \frac{1}{mn}.$$

Putting $d = \frac{1}{mn}$ in (i), we get

$$a + \frac{(m-1)}{mn} = \frac{1}{n} \Rightarrow a = \left\{\frac{1}{n} - \frac{(m-1)}{mn}\right\}$$

$$\Rightarrow a = \frac{m-(m-1)}{mn} \Rightarrow a = \frac{1}{mn}.$$

Thus, we get $a = \frac{1}{mn}$ and $d = \frac{1}{mn}$.

$\therefore \quad (mn)$th term $= a + (mn-1)d$

$$= \left\{\frac{1}{mn} + \frac{(mn-1)}{mn}\right\} = \frac{mn}{mn} = 1.$$

Hence, the (mn)th term of the given AP is 1.

THEOREM 3 *If the mth term of a given AP is n and its nth term is m then show that its pth term is* $(n+m-p)$.

PROOF In the given AP, let the first term = a and the common difference = d.

Then, $T_m = n$ and $T_n = m$

$\Rightarrow \quad a + (m-1)d = n \quad \ldots$ (i) and $a + (n-1)d = m$. $\ldots$ (ii)

On subtracting (ii) from (i), we get

$(m-n)d = (n-m) \Rightarrow d = -1$.

Putting $d = -1$ in (i), we get $a = (n+m-1)$.

$\therefore \quad p$th term $= a + (p-1)d$

$= (n+m-1) + (p-1) \times (-1)$ [$\because a = (n+m-1)$ and $d = -1$]

$= (n+m-p)$.

Hence, the pth term of the given AP is $(n+m-p)$.

THEOREM 4 *If m times the mth term of an AP is equal to n times its nth term, show that the $(m+n)$th term of the AP is zero.*

PROOF In the given AP, let the first term = *a* and the common difference = *d*.

Now, $m \cdot T_m = n \cdot T_n$

$\Rightarrow \quad m \cdot [a + (m-1)d] = n \cdot [a + (n-1)d]$

$\Rightarrow \quad [(m^2 - n^2) - (m-n)] \times d = (n-m) \times a$

$\Rightarrow \quad (m+n-1)d = -a$

$\Rightarrow \quad a + (m+n-1)d = 0$

$\Rightarrow \quad T_{m+n} = 0.$

Hence, the $(m+n)$th term of the given AP is zero.

THEOREM 5 *If the pth, qth and rth terms of an AP be a, b, c respectively, show that* $(q-r)a + (r-p)b + (p-q)c = 0.$

PROOF Let *x* be the first term and *d* be the common difference of the given AP.

Then, $T_p = a, T_q = b$ and $T_r = c$

$$\Rightarrow \begin{cases} x + (p-1)d = a & \ldots \text{(i)} \\ x + (q-1)d = b & \ldots \text{(ii)} \\ x + (r-1)d = c. & \ldots \text{(iii)} \end{cases}$$

On subtracting (ii) from (i), we get

$$(p-q)d = (a-b) \Rightarrow d = \frac{(a-b)}{(p-q)}.$$

Again, subtracting (iii) from (ii), we get

$$(q-r)d = (b-c) \Rightarrow d = \frac{(b-c)}{(q-r)}.$$

$\therefore \quad \dfrac{(a-b)}{(p-q)} = \dfrac{(b-c)}{(q-r)}$ [each equal to *d*]

$\Rightarrow \quad (q-r)a - (q-r)b = (p-q)b - (p-q)c$

$\Rightarrow \quad (q-r)a + (r-p)b + (p-q)c = 0.$

Hence, $(q-r)a + (r-p)b + (p-q)c = 0.$

REMARK The above result may also be written as

$$(a-b)r + (b-c)p + (c-a)q = 0.$$

THEOREM 6 *Show that the sum of $(m+n)$th and $(m-n)$th terms of an AP is equal to twice the mth term.*

PROOF Let *a* be the first term and *d* be the common difference of the given AP.

Then,

$$\begin{aligned} T_{m+n} + T_{m-n} &= \{a + (m+n-1)d\} + \{a + (m-n-1)d\} \\ &= 2a + 2(m-1)d \\ &= 2\{a + (m-1)d\} = 2 \times T_m. \end{aligned}$$

Hence, $T_{m+n} + T_{m-n} = 2 \times T_m.$

THEOREM 7 *Show that in an AP, the sum of the terms equidistant from the beginning and end is always equal to the sum of the first and last terms.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n$ be an AP with common difference d. Then,

(mth term from the beginning) + (mth term from the end)

$= \{a_1 + (m-1)d\} + \{a_n - (m-1)d\} = (a_1 + a_n)$

= sum of the first and last terms of the given AP.

PROBLEMS BASED ON AP

For solving problems based on AP it is always convenient to make following choices:

(i) 3 numbers in AP as $(a-d), a, (a+d)$;

(ii) 4 numbers in AP as $(a-3d), (a-d), (a+d), (a+3d)$;

(iii) 5 numbers in AP as $(a-2d), (a-d), a, (a+d), (a+2d)$;

(iv) 6 numbers in AP as $(a-5d), (a-3d), (a-d), (a+d), (a+3d), (a+5d)$.

SOLVED EXAMPLES

EXAMPLE 13 *The sum of three numbers in AP is 24 and their product is 440. Find the numbers.*

SOLUTION Let the required numbers be $(a-d), a$ and $(a+d)$. Then,

$$(a-d) + a + (a+d) = 24 \Rightarrow 3a = 24 \Rightarrow a = 8.$$

Thus, the numbers are $(8-d), 8$ and $(8+d)$.

But their product is 440.

$\therefore \quad (8-d) \times 8 \times (8+d) = 440$

$\Rightarrow \quad (8-d)(8+d) = 55$

$\Rightarrow \quad 64 - d^2 = 55 \Rightarrow d^2 = 64 - 55 = 9 \Rightarrow d = \pm 3.$

Hence, the required numbers are 5, 8, 11 or 11, 8, 5.

EXAMPLE 14 *The product of three numbers in AP is 224 and the largest number is 7 times the smallest. Find the numbers.*

SOLUTION Let the required numbers be $(a-d), a$ and $(a+d)$.

Then, largest number = $(a+d)$ and smallest number = $(a-d)$.

$$\therefore \quad a + d = 7(a-d) \Rightarrow 6a = 8d \Rightarrow d = \frac{3a}{4}. \qquad \ldots \text{(i)}$$

And, $(a-d) \times a \times (a+d) = 224$

$\Rightarrow \quad a(a^2 - d^2) = 224$

$$\Rightarrow \quad a\left(a^2 - \frac{9a^2}{16}\right) = 224 \Rightarrow a^3 = \left(224 \times \frac{16}{7}\right) = 512 = 8^3 \Rightarrow a = 8.$$

Putting $a = 8$ in (i), we get $d = \left(\frac{3}{4} \times 8\right) = 6.$

$\therefore$ the required numbers are $(8-6), 8$ and $(8+6)$, i.e., 2, 8 and 14.

EXAMPLE 15 *Find four numbers in AP whose sum is 20 and the sum of whose squares is 120.*

SOLUTION Let the required numbers be

$(a-3d), (a-d), (a+d)$ and $(a+3d)$.

Then, $(a-3d)+(a-d)+(a+d)+(a+3d)=20 \Rightarrow 4a=20 \Rightarrow a=5.$

And $(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2=120$

$\Rightarrow \quad 2(a^2+9d^2)+2(a^2+d^2)=120$

$\Rightarrow \quad 4a^2+20d^2=120 \Rightarrow a^2+5d^2=30$

$\Rightarrow \quad 25+5d^2=30 \Rightarrow 5d^2=5 \Rightarrow d^2=1 \Rightarrow d=\pm 1.$

Hence, the required numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

EXAMPLE 16 *Divide 32 into four parts which are in AP such that the product of extremes is to the product of means as 7 : 15.*

SOLUTION Let the required parts be $(a-3d), (a-d), (a+d)$ and $(a+3d)$.

Then, $(a-3d)+(a-d)+(a+d)+(a+3d)=32 \Rightarrow 4a=32 \Rightarrow a=8.$

So, the required parts are $(8-3d), (8-d), (8+d)$ and $(8+3d)$.

But product of extremes is to the product of means as 7 : 15.

$\therefore \quad \dfrac{(8-3d)(8+3d)}{(8-d)(8+d)}=\dfrac{7}{15}$

$\Rightarrow \quad \dfrac{(64-9d^2)}{(64-d^2)}=\dfrac{7}{15}$

$\Rightarrow \quad 15(64-9d^2)=7(64-d^2)$

$\Rightarrow \quad 128d^2=512 \Rightarrow d^2=\dfrac{512}{128}=4 \Rightarrow d=\pm 2.$

Thus, $(a=8 \text{ and } d=2)$ or $(a=8 \text{ and } d=-2)$.

Hence, the required parts are 2, 6, 10, 14 or 14, 10, 6, 2.

EXERCISE 11A

1. Write first 4 terms in each of the sequences:

(i) $a_n=(5n+2)$ (ii) $a_n=\dfrac{(2n-3)}{4}$ (iii) $a_n=(-1)^{n-1}\times 2^{n+1}$

2. Find first five terms of the sequence, defined by

$a_1=1, a_n=a_{n-1}+3$ for $n\geq 2.$

3. Find first 5 terms of the sequence, defined by

$a_1=-1, a_n=\dfrac{a_{n-1}}{n}$ for $n\geq 2.$

4. Find the 23rd term of the AP 7, 5, 3, 1, –1, –3, … .

5. Find the 20th term of the AP $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, \ldots$.

6. Find the *n*th term of the AP 8, 3, –2, –7, –12, … .

7. Find the *n*th term of the AP $1, \dfrac{5}{6}, \dfrac{2}{3}, \dfrac{1}{2}, \ldots$.

8. Which term of the AP 9, 14, 19, 24, 29, … is 379?
9. Which term of the AP 64, 60, 56, 52, 48, … is 0?
10. How many terms are there in the AP 11, 18, 25, 32, 39, …, 207?
11. How many terms are there in the AP $1\frac{5}{6}, 1\frac{1}{6}, \frac{1}{2}, \frac{-1}{6}, \frac{-5}{6}, \ldots, -16\frac{1}{6}$?
12. Is –47 a term of the AP 5, 2, −1, − 4, −7, …?
13. The 5th and 13th terms of an AP are 5 and –3 respectively. Find the AP and its 30th term.
14. The 2nd, 31st and the last terms of an AP are $7\frac{3}{4}, \frac{1}{2}$ and $-6\frac{1}{2}$ respectively. Find the first term and the number of terms.
15. If the 9th term of an AP is 0, prove that its 29th term is double the 19th term.
16. The 4th term of an AP is three times the first and the 7th term exceeds twice the third term by 1. Find the first term and the common difference.
17. If 7 times the 7th term of an AP is equal to 11 times its 11th term, show that the 18th term of the AP is zero.
18. Find the 28th term from the end of the AP 6, 9, 12, 15, 18, …, 102.
19. Find the 16th term from the end of the AP 7, 2, −3, −8, −13, …, −113.
20. How many 3-digit numbers are divisible by 7?
21. How many 2-digit numbers are divisible by 3?
22. If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP whose common difference is d, show that
$$\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n = \frac{(\tan\theta_n - \tan\theta_1)}{\sin d}.$$
23. In an AP, it is being given that $\frac{T_4}{T_7} = \frac{2}{3}$. Find $\frac{T_7}{T_{10}}$.
24. Three numbers are in AP. If their sum is 27 and their product is 648, find the numbers.
25. The sum of three consecutive terms of an AP is 21 and the sum of the squares of these terms is 165. Find these terms.
26. The angles of a quadrilateral are in AP whose common difference is 10°. Find the angles.
27. The digits of a 3-digit number are in AP and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.
28. Find the number of terms common to the two arithmetic progressions 5, 9, 13, 17, …, 217 and 3, 9, 15, 21, …, 321.
29. We know that the sum of the interior angles of a triangle is 180°. Show that the sum of the interior angles of polygons with 3, 4, 5, 6, … sides form an arithmetic progression. Find the sum of the interior angles for a 21-sided polygon.
30. A side of an equilateral triangle is 24 cm long. A second equilateral triangle is inscribed in it by joining the midpoints of the sides of the first triangle. The

process is continued. Find the perimeter of the sixth inscribed equilateral triangle.

31. A man starts repaying a loan as the first instalment of ` 10000. If he increases the instalments by ` 500 every month, what amount will he pay in 30th instalment?

ANSWERS (EXERCISE 11A)

1. (i) 7, 12, 17, 22 (ii) $\frac{-1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}$ (iii) 4, –8, 16, –32 **2.** 1, 4, 7, 10, 13

3. $-1, \frac{-1}{2}, \frac{-1}{6}, \frac{-1}{24}, \frac{-1}{120}$ **4.** –37 **5.** $39\sqrt{2}$ **6.** $T_n = (13 - 5n)$

7. $T_n = \frac{1}{6}(7 - n)$ **8.** 75th **9.** 17th **10.** 29 **11.** 28

12. No **13.** $(9, 8, 7, 6, \ldots), T_{30} = -20$ **14.** $a = 8, n = 59$

16. $a = 3, d = 2$ **18.** 21 **19.** –38 **20.** 128

21. 30 **23.** $\frac{3}{4}$ **24.** 6, 9, 12 **25.** 4, 7, 10

26. $75°, 85°, 95°, 105°$ **27.** 852 **28.** 18 **29.** 3420°

30. 2.25 cm **31.** ` 24500

HINTS TO SOME SELECTED QUESTIONS

14. Let a be the first term and d be the common difference of the given AP.

Then, $T_2 = \frac{31}{4} \Rightarrow a + d = \frac{31}{4} \Rightarrow 4a + 4d = 31$ … (i)

and $T_{31} = \frac{1}{2} \Rightarrow a + 30d = \frac{1}{2} \Rightarrow 2a + 60d = 1.$ … (ii)

On solving (i) and (ii), we get $a = 8$ and $d = \frac{-1}{4}$.

Let $T_n = \frac{-13}{2}$. Then, $a + (n-1)d = \frac{-13}{2} \Rightarrow 8 + (n-1) \times \left(\frac{-1}{4}\right) = \frac{-13}{2} \Rightarrow n = 59.$

15. $T_9 = 0 \Rightarrow a + 8d = 0 \Rightarrow a = -8d.$

$T_{29} = a + 28d = -8d + 28d = 20d.$

$T_{19} = a + 18d = -8d + 18d = 10d.$

Hence, $T_{29} = 2 \times T_{19}$.

16. $T_4 = 3a$ and $T_7 - 2T_3 = 1$

$\Rightarrow \quad a + 3d = 3a$ and $(a + 6d) - 2(a + 2d) = 1$

$\Rightarrow \quad 2a - 3d = 0$ … (i) and $-a + 2d = 1.$ … (ii)

On solving (i) and (ii), we get $a = 3$ and $d = 2$.

20. The given numbers are 105, 112, 119, …, 994.

This is an AP for which $a = 105$, $d = 7$ and $l = 994$.

Let $T_n = 994$. Then, $105 + (n-1) \times 7 = 994 \Rightarrow n = 128.$

22. Let $\theta_2 - \theta_1 = \theta_3 - \theta_2 = \theta_4 - \theta_3 = \ldots = \theta_n - \theta_{n-1} = d.$

$\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n$

$$= \frac{1}{\sin d}\left\{\frac{\sin d}{\cos\theta_1\cos\theta_2} + \frac{\sin d}{\cos\theta_2\cos\theta_3} + \ldots + \frac{\sin d}{\cos\theta_{n-1}\cos\theta_n}\right\}$$

$$= \frac{1}{\sin d}\cdot\left\{\frac{\sin(\theta_2-\theta_1)}{\cos\theta_1\cos\theta_2} + \frac{\sin(\theta_3-\theta_2)}{\cos\theta_2\cos\theta_3} + \ldots + \frac{\sin(\theta_n-\theta_{n-1})}{\cos\theta_{n-1}\cos\theta_n}\right\}$$

$$= \frac{1}{\sin d}\cdot\left\{\frac{\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1}{\cos\theta_1\cos\theta_2} + \frac{\sin\theta_3\cos\theta_2 - \cos\theta_3\sin\theta_2}{\cos\theta_2\cos\theta_3} + \right.$$

$$\left. \ldots + \frac{\sin\theta_n\cos\theta_{n-1} - \cos\theta_n\sin\theta_{n-1}}{\cos\theta_{n-1}\cos\theta_n}\right\}$$

$$= \frac{1}{\sin d}\cdot\{(\tan\theta_2 - \tan\theta_1) + (\tan\theta_3 - \tan\theta_2) + \ldots + (\tan\theta_n - \tan\theta_{n-1})\}$$

$$= \frac{1}{\sin d}\cdot(\tan\theta_n - \tan\theta_1) = \frac{(\tan\theta_n - \tan\theta_1)}{\sin d}.$$

23. $\frac{a+3d}{a+6d} = \frac{2}{3} \Rightarrow 3a+9d = 2a+12d \Rightarrow a = 3d.$

$\therefore \quad \frac{T_7}{T_{10}} = \frac{a+6d}{a+9d} = \frac{3d+6d}{3d+9d} = \frac{9d}{12d} = \frac{3}{4}.$

26. Let the required angles be $a^\circ, (a+10)^\circ, (a+20)^\circ, (a+30)^\circ$. Then,

$a + a + 10 + a + 20 + a + 30 = 360 \Rightarrow 4a = 300 \Rightarrow a = 75.$

27. Let the hundred's, ten's and unit digit's be $a+d, a, a-d$.

Then, $a + d + a + a - d = 15 \Rightarrow 3a = 15 \Rightarrow a = 5.$

$[100(a+d) + 10a + (a-d)] - [100(a-d) + 10a + (a+d)] = 594$

$\Rightarrow \quad 99d + 99d = 594 \Rightarrow 198d = 594 \Rightarrow d = 3.$

Hence, the required number is 852.

28. The first common term in two APs is 9.

LCM of their common differences = LCM (4, 6) = 12.

New AP of common elements is 9, 21, 33, 45, … .

Let it contain n terms. Then,

$T_n \le 216 \Rightarrow 9 + (n-1)\times 12 \le 216 \Rightarrow (n-1)\times 12 \le 207$

$\Rightarrow (n-1)\times 12 = 204 \Rightarrow n = 18.$

So, there are 18 common terms in two APs.

29. The sum of the interior angles of a triangle is 180°. A polygon of 4 sides can be divided into two triangles and therefore, the sum of its angles is 360°. A polygon of 5 sides can be divided into 3 triangles and therefore, the sum of its angles is $(3\times 180^\circ) = 540^\circ$ and so on. Thus, a polygon of 21 sides is divided into $(21-2) = 19$ triangles and therefore, the sum of its angles is $(19\times 180^\circ) = 3420^\circ$.

30. We know that the line segment joining midpoints of the opposite sides of a triangle is parallel to the third side and equal to half of it.

Perimeters of 1st, 2nd, 3rd, 4th, 5th and 6th triangle are respectively 72 cm, 36 cm, 18 cm, 9 cm, $\frac{9}{2}$ cm, $\frac{9}{4}$ cm.

31. The instalments form the AP

10000 + 10500 + 11000 + 11500 + … up to 30 terms.

Here $a = 10000$, $d = 500$ and we have to find T_{30}.

SUM OF *n* TERMS OF AN AP

THEOREM 1 *Prove that the sum of n terms of an AP with first term a and common difference d is given by:*

I. $S_n = \frac{n}{2}(a + l)$, *where l is the last term.*

II. $S_n = \frac{n}{2} \cdot [2a + (n-1)d]$.

PROOF Let us consider an AP containing *n* terms with the first term *a*, the common difference *d* and the last term *l*.

Let the sum of these *n* terms be S_n. Then,

$$S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l. \quad \ldots \text{(i)}$$

Writing the above series in a reverse order, we get

$$S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a. \quad \ldots \text{(ii)}$$

Adding the corresponding terms of (i) and (ii), we get

$$2S_n = \{(a + l) + (a + l) + \ldots \text{to } n \text{ times}\} = n(a + l).$$

$$\therefore \quad S_n = \frac{n}{2}(a + l).$$

But, $l = n$th term $= a + (n-1)d$.

$$\therefore \quad S_n = \frac{n}{2} \cdot \{a + a + (n-1)d\} = \frac{n}{2} \cdot \{2a + (n-1)d\}.$$

Hence, $S_n = \frac{n}{2} \cdot \{2a + (n-1)d\}$.

SUMMARY

(i) Sum of *n* terms of an AP with the first term $= a$ and the last term $= l$ is given by

$$S_n = \frac{n}{2}(a + l).$$

(ii) Sum of *n* terms of an AP with the first term $= a$ and the common difference $= d$ is given by

$$S_n = \frac{n}{2} \cdot \{2a + (n-1)d\}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the sum of* 23 *terms of the AP* 5, 9, 13, 17, … .

SOLUTION Here, $a = 5$, $d = (9 - 5) = 4$ and $n = 23$.

Now, $S_n = \frac{n}{2} \times \{2a + (n-1) \times d\}$.

$$\therefore \quad S_{23} = \frac{23}{2} \times \{2 \times 5 + (23 - 1) \times 4\}$$

$$= \left(\frac{23}{2} \times 98\right) = 1127.$$

Hence, the sum of 23 terms of the given AP is 1127.

EXAMPLE 2 *The first, second and the last terms of an AP are a, b, c respectively. Prove that the sum of the AP is* $\frac{(a+c)(b+c-2a)}{2(b-a)}$.

SOLUTION Let *d* be the common difference of the given AP and let it contain *n* terms. Then, $d = (b-a)$.

Now, $T_n = c \Rightarrow a+(n-1)d = c$

$\Rightarrow a+(n-1)(b-a) = c \qquad [\because\ d = (b-a)]$

$\Rightarrow (n-1) = \frac{(c-a)}{(b-a)}$

$\Rightarrow n = \left\{\frac{(c-a)}{(b-a)}+1\right\} = \frac{(b+c-2a)}{(b-a)}$.

$\therefore$ sum of the given AP $= \frac{n}{2}(a+l)$, where *l* is the last term

$= \frac{(b+c-2a)}{2(b-a)} \times (a+c)$

$\left[\because\ n = \frac{(b+c-2a)}{2(b-a)} \text{ and } l = c\right]$

$= \frac{(a+c)(b+c-2a)}{2(b-a)}$.

EXAMPLE 3 *How many terms of the AP* $-6, \frac{-11}{2}, -5, \frac{-9}{2}, \ldots$ *are needed to give the sum* -25*? Explain the double answer.*

SOLUTION Let the required number of terms be *n*. Then,

$a = -6,\ d = \left\{\frac{-11}{2} - (-6)\right\} = \left(\frac{-11}{2}+6\right) = \frac{1}{2}$ and $S_n = -25$.

$\therefore\ S_n = -25 \Rightarrow \frac{n}{2}\cdot[2a+(n-1)d] = -25$

$\Rightarrow \frac{n}{2}\cdot\left[2\times(-6)+(n-1)\cdot\frac{1}{2}\right] = -25$

$\Rightarrow \frac{n}{2}\cdot\left(\frac{n}{2}-\frac{25}{2}\right) = -25 \Rightarrow \frac{n}{2}\cdot\frac{(n-25)}{2} = -25$

$\Rightarrow n^2 - 25n + 100 = 0 \Rightarrow (n-5)(n-20) = 0$

$\Rightarrow n = 5$ or $n = 20$.

This shows that the sum of first five terms is –25 and that the sum of first 20 terms is also –25.

It means that the sum of all the terms from 6th to 20th is zero.

EXAMPLE 4 *Find the value of x when* $1+6+11+16+\ldots+x = 148$.

SOLUTION The given series is an arithmetic series in which $a = 1$ and $d = (6-1) = 5$.

Let it contain *n* terms. Then, we have to find the *n*th term.

Now, $S_n = 148$

$\Rightarrow \quad \frac{n}{2} \cdot [2a + (n-1)d] = 148$

$\Rightarrow \quad \frac{n}{2} \cdot [2 \times 1 + (n-1) \times 5] = 148$

$\Rightarrow \quad n(5n-3) = 296 \Rightarrow 5n^2 - 3n - 296 = 0$

$\Rightarrow \quad 5n^2 - 40n + 37n - 296 = 0$

$\Rightarrow \quad 5n(n-8) + 37(n-8) = 0 \Rightarrow (n-8)(5n+37) = 0$

$\Rightarrow \quad n = 8 \quad \left[\because n \neq \frac{-37}{5}\right].$

$\therefore \quad T_8 = 1 + (8-1) \times 5 = 36 \quad [\because T_n = a + (n-1)d].$

Hence, $x = 36$.

EXAMPLE 5 *Find the sum of all odd integers from* 1 *to* 1001.

SOLUTION The odd integers from 1 to 1001 are 1, 3, 5, 7, …, 999, 1001.

This is an AP in which $a = 1$, $d = (3-1) = 2$ and $l = 1001$.

Let the number of terms be n. Then,

$T_n = 1001 \Rightarrow a + (n-1)d = 1001$

$\Rightarrow 1 + (n-1) \times 2 = 1001 \Rightarrow n = 501.$

Now, $a = 1$, $l = 1001$ and $n = 501$.

$\therefore \quad S_n = \frac{n}{2}(a+l) = \frac{501}{2} \cdot (1 + 1001) = (501 \times 501) = 251001.$

Hence, the required sum is 251001.

EXAMPLE 6 *Find the sum of all natural numbers lying between* 100 *and* 1000, *which are multiples of* 5.

SOLUTION Required numbers are 105, 110, 115, …, 995.

This is an AP in which $a = 105$, $d = (110 - 105) = 5$ and $l = 995$.

Let the number of terms in this AP be n. Then,

$T_n = 995 \Rightarrow a + (n-1)d = 995$

$\Rightarrow 105 + (n-1) \times 5 = 995$

$\Rightarrow (n-1) = \frac{890}{5} = 178 \Rightarrow n = 179.$

$\therefore \quad S_n = \frac{n}{2}(a+l) = \frac{179}{2} \times (105 + 995) = \left(\frac{179}{2} \times 1100\right)$

$= (179 \times 550) = 98450.$

Hence, the required sum is 98450.

EXAMPLE 7 *Find the sum of integers from* 1 *to* 100 *that are divisible by* 2 *or* 5.

SOLUTION Required sum

= (sum of integers from 1 to 100, divisible by 2)

+ (sum of integers from 1 to 100, divisible by 5)

– (sum of integers from 1 to 100, divisible by 10)

$$= (2+4+6+\ldots+100)+(5+10+15+\ldots+100)$$
$$-(10+20+30+\ldots+100)$$
$$= \frac{50}{2}(2+100)+\frac{20}{2}(5+105)-\frac{10}{2}(10+100)$$
$$= (2550+1050-550) = 3050.$$

EXAMPLE 8 *Find the sum of all two-digit numbers which when divided by* 4 *yield* 1 *as remainder.*

SOLUTION Required sum

$$13+17+21+25+\ldots+97.$$

This is an AP in which $a = 13$, $d = (17-13) = 4$ and $l = 97$.

Let the number of terms be n. Then,

$$T_n = 97 \Rightarrow a+(n-1)d = 97$$
$$\Rightarrow 13+(n-1)\times 4 = 97 \Rightarrow n = 22.$$

$$\therefore \quad \text{required sum} = \frac{n}{2}(a+l) = \frac{22}{2}\times(13+97) = (11\times 110) = 1210.$$

Hence, the required sum is 1210.

EXAMPLE 9 *The sum of n terms of two arithmetic progressions are in the ratio* $(3n+8):(7n+15)$. *Find the ratio of their* 12*th terms.*

SOLUTION Let a_1, a_2 and d_1, d_2 be the first terms and common differences of the first and second AP respectively. Then, by the given condition, we have

$$\frac{\text{sum to } n \text{ terms of first AP}}{\text{sum to } n \text{ terms of second AP}} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{\frac{n}{2}\cdot[2a_1+(n-1)d_1]}{\frac{n}{2}\cdot[2a_2+(n-1)d_2]} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2} = \frac{3n+8}{7n+15}. \qquad \ldots \text{(i)}$$

Now, $\frac{12\text{th term of first AP}}{12\text{th term of second AP}} = \frac{a_1+11d_1}{a_2+11d_2}$

$$= \frac{2a_1+22d_1}{2a_2+22d_2} = \frac{2a_1+(23-1)d_1}{2a_2+(23-1)d_2}$$
$$= \frac{3\times 23+8}{7\times 23+15} \quad [\text{putting } n = 23 \text{ in (i)}]$$
$$= \frac{77}{176} = \frac{7}{16}.$$

Hence, the ratio of the 12th terms of given APs is 7 : 16.

EXAMPLE 10 *In an AP, the first term is* 2 *and the sum of first five terms is one-fourth of the sum of next five terms. Show that its* 20*th term is* –112 *and the sum of its first* 20 *terms is* –1100.

SOLUTION We have

$$(T_1 + T_2 + T_3 + T_4 + T_5) = \frac{1}{4}(T_6 + T_7 + T_8 + T_9 + T_{10})$$

$$= \frac{1}{4} \cdot [(T_1 + T_2 + \ldots + T_5 + T_6 + \ldots + T_{10}) - (T_1 + T_2 + T_3 + T_4 + T_5)]$$

$\Rightarrow \quad S_5 = \frac{1}{4} \cdot (S_{10} - S_5)$, where S_n denotes the sum of first n terms

$\Rightarrow \quad 4S_5 = S_{10} - S_5$

$\Rightarrow \quad 5S_5 = S_{10}$

$\Rightarrow \quad 5 \times \frac{5}{2} \times [2 \times 2 + (5-1)d] = \frac{10}{2} \times [2 \times 2 + (10-1)d]$

$\Rightarrow \quad 50(1 + d) = (20 + 45d)$

$\Rightarrow \quad 5d = -30 \Rightarrow d = -6.$

$\therefore \quad a = 2$ and $d = -6.$

$\therefore \quad T_{20} = [2 + (20-1) \times (-6)] = -112$

and $\quad S_{20} = \frac{20}{2} \times [2 \times 2 + (20-1) \times (-6)] = -1100.$

EXAMPLE 11 *The difference between any two consecutive interior angles of a polygon is* 5°. *If the smallest angle is* 120°, *find the number of sides of the polygon.*

SOLUTION Let the number of sides of the polygon be n.

Then, the sum of its interior angles = $(2n - 4)$ right angles

$$= \{(n-2) \times 180\}^\circ. \qquad \ldots \text{(i)}$$

Since the difference between any two consecutive interior angles of the polygon is constant, its angles are in AP.

In this AP, we have $a = 120$ and $d = 5$.

Now, $S_n = (n-2) \times 180$ [from (i)]

$\Rightarrow \quad \frac{n}{2} \cdot \{2 \times 120 + (n-1) \times 5\} = (n-2) \times 180$

$\Rightarrow \quad \frac{235n}{2} + \frac{5n^2}{2} = 180n - 360 \Rightarrow 5n^2 - 125n + 720 = 0$

$\Rightarrow \quad n^2 - 25n + 144 = 0 \Rightarrow (n-9)(n-16) = 0$

$\Rightarrow \quad n = 9$ or $n = 16.$

But, when $n = 16$, we have

last angle = $\{120 + (16-1) \times 5\}^\circ = 195^\circ$, which is not possible.

$\therefore \quad n = 9.$

Hence, the number of sides of the given polygon = 9.

EXAMPLE 12 *The income of a man is* ` 400000 *in the first year and he receives an increase of* ` 10000 *to his income per year for* 19 *years. Find the total amount he received in* 20 *years.*

SOLUTION Here, we have an AP with $a = 400000$, $d = 10000$ and $n = 20$.

Using the formula, $S_n = \frac{n}{2} \times [2a + (n-1)d]$, we have

$$S_{20} = \frac{20}{2} \times [2 \times 400000 + 19 \times 10000] = 10 \times 990000 = 9900000.$$

Hence, the man received ` 9900000 as the total amount in 20 years.

EXAMPLE 13 *A man saved* ` 660000 *in* 20 *years. In each succeeding year after the first year he saves* ` 2000 *more than what he saved in the previous year. How much did he save in the first year?*

SOLUTION Here, we have an AP in which $S_{20} = 660000$, $d = 2000$ and we have to find the value of a.

Using the formula, $S_n = \frac{n}{2} \times [2a + (n-1)d]$, we get

$$S_{20} = \frac{20}{2} \times [2a + (20-1) \times 2000]$$

$$\Rightarrow \quad 10 \times [2a + 19 \times 2000] = 660000 \qquad [\because\ S_{20} = 660000]$$

$$\Rightarrow \quad 20a + 380000 = 660000 \ \Rightarrow\ 20a = 280000 \ \Rightarrow\ a = 14000.$$

Hence, the man saved ` 14000 in the first year.

EXAMPLE 14 *In a potato race,* 20 *potatoes are placed in a line at intervals of* 4 *metres with the first potato* 24 *metres from the starting point. A contestant is required to bring the potatoes back to the starting place, one at a time. How far would he run in bringing back all the potatoes?*

SOLUTION Distances moved in bringing back the 1st, 2nd, 3rd, …, 20th potato to the starting point are (2×24) m, (2×28) m, (2×32) m, …, and so on.

Total distance moved = $\{48 + 56 + 64 + 72 + \ldots$ to 20 terms$\}$

$$= \frac{20}{2} \times [2 \times 48 + (20-1) \times 8] \text{ m}$$

$$[\because\ a = 48, d = 8 \text{ and } n = 20]$$

$$= 10 \times (96 + 152) \text{ m} = 2480 \text{ m}.$$

Hence, total distance covered = 2480 m.

EXAMPLE 15 *In a cricket tournament* 16 *school teams participated. A sum of* ` 16000 *is to be awarded among themselves as prize money. If the team in the last place is awarded* ` 550 *in prize money and the award increases by the same amount for successive finishing places, how much amount will the team in the first place receive?*

SOLUTION Let the first place team receive ` $a$ and let the award increase by ` d. Then, these prizes form an AP in which first term = a, common difference = d, $T_{16} = 550$ and $S_{16} = 16000$.

Now, $T_{16} = 550 \ \Rightarrow\ a + 15d = 550.$ … (i)

And, $S_{16} = 16000 \ \Rightarrow\ \frac{16}{2} \times [2a + 15d] = 16000$

$$\Rightarrow\ 2a + 15d = 2000. \qquad \ldots \text{(ii)}$$

On subtracting (i) from (ii), we get $a = 1450$.

Hence, the team in the first place receives ` 1450.

EXAMPLE 16 *The first term of an AP is a and the sum of the first p terms is zero. Show that the sum of the next q terms is* $\frac{-a(p+q)q}{(p-1)}$.

SOLUTION Let the common difference of the given AP be *d*. Then,

$$S_p = 0 \Rightarrow \frac{p}{2} \times [2a + (p-1)d] = 0$$

$$\Rightarrow 2a + (p-1)d = 0 \Rightarrow d = \frac{-2a}{(p-1)}. \quad \ldots \text{(i)}$$

$$\therefore \quad S_{p+q} = \frac{(p+q)}{2} \cdot [2a + (p+q-1)d]$$

$$= \frac{(p+q)}{2} \cdot \left[2a + (p+q-1) \times \frac{(-2a)}{(p-1)}\right] \quad \text{[using (i)]}$$

$$= a(p+q) \cdot \left[1 - \frac{(p+q-1)}{(p-1)}\right] = \frac{-a(p+q)q}{(p-1)}.$$

Hence, the required sum is $\frac{-a(p+q)q}{(p-1)}$.

EXAMPLE 17 *If the mth term of an AP is a and its nth term is b, show that the sum of its* $(m+n)$ *terms is* $\frac{(m+n)}{2} \cdot \left\{a + b + \frac{(a-b)}{(m-n)}\right\}$.

SOLUTION Let the first term and the common difference of the given AP be *A* and *d* respectively. Then,

$$T_m = a \Rightarrow A + (m-1)d = a \quad \ldots \text{(i)}$$

and $T_n = b \Rightarrow A + (n-1)d = b.$ $\ldots$ (ii)

On subtracting (ii) from (i), we get

$$(m-n)d = (a-b) \Rightarrow d = \frac{(a-b)}{(m-n)}. \quad \ldots \text{(iii)}$$

Adding (i) and (ii), we get

$$2A + (m+n-2)d = a + b$$

$$\Rightarrow \quad 2A + (m+n-1)d - d = a + b$$

$$\Rightarrow \quad 2A + (m+n-1)d = (a+b+d) = \left(a + b + \frac{a-b}{m-n}\right) \quad \ldots \text{(iv)}$$

[using (iii)].

$$\therefore \quad T_{m+n} = \frac{(m+n)}{2} \cdot [2A + (m+n-1)d].$$

$$\Rightarrow \quad T_{m+n} = \frac{(m+n)}{2} \cdot \left[a + b + \frac{a-b}{m-n}\right] \quad \text{[using (iv)].}$$

EXAMPLE 18 *If the ratio of the sums of m and n terms of an AP be* $(m^2 : n^2)$, *show that the ratio of their mth and nth terms is* $(2m-1) : (2n-1)$.

SOLUTION Let the first term and the common difference of the given AP be *a* and *d* respectively. Then,

$$\frac{S_m}{S_n}=\frac{m^2}{n^2} \Rightarrow \frac{\frac{m}{2}\cdot[2a+(m-1)d]}{\frac{n}{2}\cdot[2a+(n-1)d]}=\frac{m^2}{n^2}$$

$$\Rightarrow \frac{2a+(m-1)d}{2a+(n-1)d}=\frac{m}{n}$$

$$\Rightarrow n[2a+(m-1)d]=m[2a+(n-1)d]$$

$$\Rightarrow 2a(m-n)=d(m-n) \Rightarrow d=2a.$$

$$\therefore \quad \frac{T_m}{T_n}=\frac{a+(m-1)d}{a+(n-1)d}=\frac{a+2(m-1)a}{a+2(n-1)a}=\frac{1+2m-2}{1+2n-2}=\frac{2m-1}{2n-1}.$$

Hence, the ratio of their mth and nth terms is $(2m-1):(2n-1)$.

SOME RESULTS ON SUMMATION OF AP

THEOREM 1 *If the sum of first n terms of a progression is a quadratic expression in n, show that it is an AP.*

PROOF Let $S_n=an^2+bn+c$ … (i), where a, b, c are constants and $a\neq 0$.

On replacing n by $(n-1)$ in (i), we get

$$S_{n-1}=a(n-1)^2+b(n-1)+c.$$

$$\therefore \quad T_n=(S_n-S_{n-1})$$

$$=(an^2+bn+c)-\{a(n-1)^2+b(n-1)+c\}$$

$$=a\{n^2-(n-1)^2\}+b\{n-(n-1)\}$$

$$=a(2n-1)+b=2an+(b-a).$$

Thus, $T_n=2an+(b-a)$. … (ii)

Replacing n by $(n-1)$ in (ii), we get

$$T_{n-1}=2a(n-1)+(b-a). \quad \text{… (iii)}$$

On subtracting (iii) from (ii), we get

$(T_n-T_{n-1})=2a$, which is constant.

Hence, the given progression is an AP.

THEOREM 2 *If there are $(2n+1)$ terms in an AP, prove that the sum of odd terms and the sum of even terms bear the ratio $(n+1):n$.*

PROOF Since the given AP contains $(2n+1)$ terms, so it has $(n+1)$ odd terms and n even terms.

Let a be the first term and d be the common difference of the given AP.

Let S_1 and S_2 be the sum of all odd terms and the sum of all even terms respectively. Then,

$$S_1=T_1+T_3+T_5+\ldots+T_{2n+1}$$

$$=\frac{(n+1)}{2}\cdot[T_1+T_{2n+1}] \qquad \text{[sum of } (n+1) \text{ odd terms]}$$

$$=\frac{(n+1)}{2}\cdot[a+\{a+(2n+1-1)d\}] \qquad [\because\ T_{2n+1}=a+(2n+1-1)d]$$

$$=(n+1)(a+nd).$$

$$S_2 = T_2 + T_4 + T_6 + \ldots + T_{2n}$$

$$= \frac{n}{2} \cdot (T_2 + T_{2n}) \qquad \text{[sum of } n \text{ even terms]}$$

$$= \frac{n}{2} \cdot [(a+d) + \{a + (2n-1)d\}] \qquad [\because\ T_{2n} = a + (2n-1)d]$$

$$= n(a + nd).$$

$$\therefore \quad \frac{S_1}{S_2} = \frac{(n+1)(a+nd)}{n(a+nd)} \Rightarrow \frac{S_1}{S_2} = \frac{n+1}{n}.$$

Hence, the required ratio is $(n+1) : n$.

THEOREM 3 *If the mth term of an AP is $\left(\frac{1}{n}\right)$ and its nth term is $\left(\frac{1}{m}\right)$ then show that the sum of its mn terms is $\frac{1}{2}(mn+1)$.*

PROOF Let a be the first term and d be the common difference of the given AP.

Then, $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$.

$$\therefore \quad a + (m-1)d = \frac{1}{n} \quad \ldots \text{(i)} \qquad \text{and} \quad a + (n-1)d = \frac{1}{m}. \quad \ldots \text{(ii)}$$

On subtracting (ii) from (i), we get

$$(m-n)d = \left(\frac{1}{n} - \frac{1}{m}\right) = \frac{(m-n)}{mn} \Rightarrow d = \frac{1}{mn}.$$

Putting $d = \frac{1}{mn}$ in (i), we get

$$a + \frac{(m-1)}{mn} = \frac{1}{n} \Rightarrow a = \left\{\frac{1}{n} - \frac{(m-1)}{mn}\right\} = \frac{1}{mn}.$$

$$\therefore \quad S_{mn} = \frac{mn}{2} \cdot [2a + (mn-1)d]$$

$$= \frac{mn}{2} \cdot \left\{\frac{2}{mn} + \frac{(mn-1)}{mn}\right\} \qquad \left[\because\ a = \frac{1}{mn} \text{ and } d = \frac{1}{mn}\right]$$

$$= \frac{1}{2}(mn+1).$$

Hence, the sum of mn terms of the given AP is $\frac{1}{2}(mn+1)$.

THEOREM 4 *If the sum of first p terms of an AP is the same as the sum of its first q terms then show that the sum of its first $(p+q)$ terms is zero.*

PROOF Let a be the first term and d be the common difference of the given AP.

Then, $S_p = S_q$

$$\Rightarrow \quad \frac{p}{2} \cdot [2a + (p-1)d] = \frac{q}{2} \cdot [2a + (q-1)d]$$

$$\Rightarrow \quad 2(p-q)a + \{p(p-1) - q(q-1)\}d = 0$$

$$\Rightarrow \quad 2(p-q)a + \{(p^2 - q^2) - (p-q)\}d = 0$$

$$\Rightarrow \quad 2(p-q)a + (p-q)(p+q-1)d = 0$$

$$\Rightarrow \quad (p-q)\{2a + (p+q-1)d\} = 0$$

$$\Rightarrow \quad 2a + (p+q-1)d = 0. \quad \ldots \text{(i)}$$

$$\therefore \quad S_{p+q} = \frac{(p+q)}{2} \cdot [2a + (p+q-1)d] = 0 \qquad \text{[using (i)].}$$

Hence, the sum of the first $(p+q)$ term is zero.

THEOREM 5 *If the sum of m terms of an AP be n and the sum of its n terms be m, show that the sum of its $(m+n)$ terms is $-(m+n)$.*

PROOF Let *a* be the first term and *d* be the common difference of the given AP.

Then, $S_m = n \Rightarrow \frac{m}{2} \cdot \{2a + (m-1)d\} = n$

$$\Rightarrow 2am + m(m-1)d = 2n. \qquad \ldots \text{(i)}$$

And, $S_n = m \Rightarrow \frac{n}{2} \cdot \{2a + (n-1)d\} = m$

$$\Rightarrow 2an + n(n-1)d = 2m. \qquad \ldots \text{(ii)}$$

On subtracting (ii) from (i), we get

$$2a(m-n) + \{(m^2 - n^2) - (m-n)\} \cdot d = 2(n-m)$$

$$\Rightarrow \quad 2a(m-n) + (m-n)(m+n-1)d = 2(n-m)$$

$$\Rightarrow \quad (m-n)\{2a + (m+n-1)d\} = -2(m-n)$$

$$\Rightarrow \quad 2a + (m+n-1)d = -2. \qquad \ldots \text{(iii)}$$

$$\therefore \quad S_{m+n} = \frac{(m+n)}{2} \cdot \{2a + (m+n-1)d\}$$

$$= \frac{(m+n)}{2} \cdot (-2) = -(m+n) \qquad \text{[using (iii)].}$$

Hence, the sum of $(m+n)$ terms is $-(m+n)$.

THEOREM 6 *The sum of the first p, q and r terms of an AP are a, b, c respectively. Show that*

$$\frac{a}{p} \cdot (q-r) + \frac{b}{q} \cdot (r-p) + \frac{c}{r} \cdot (p-q) = 0.$$

PROOF Let *x* be the first term and *d* be the common difference of the given AP.

Then, $S_p = a$, $S_q = b$ and $S_r = c$.

Now, $S_p = a \Rightarrow \frac{p}{2} \cdot [2x + (p-1)d] = a$

$$\Rightarrow \frac{a}{p} = x + (p-1) \cdot \frac{d}{2} \cdot \qquad \ldots \text{(i)}$$

$$S_q = b \Rightarrow \frac{q}{2} \cdot [2x + (q-1)d] = b$$

$$\Rightarrow \frac{b}{q} = x + (q-1) \cdot \frac{d}{2} \cdot \qquad \ldots \text{(ii)}$$

And, $S_r = c \Rightarrow \frac{r}{2} \cdot [2x + (r-1)d] = c$

$$\Rightarrow \frac{c}{r} = x + (r-1) \cdot \frac{d}{2} \cdot \qquad \ldots \text{(iii)}$$

On multiplying (i), (ii) and (iii) by $(q-r)$, $(r-p)$ and $(p-q)$ respectively and adding, we get

$$\frac{a}{p}\cdot(q-r)+\frac{b}{q}\cdot(r-p)+\frac{c}{r}\cdot(p-q)$$
$$=x\cdot[(q-r)+(r-p)+(p-q)]$$
$$+\frac{d}{2}\cdot[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]$$
$$=(x\times 0)+\left(\frac{d}{2}\times 0\right)=0.$$

Hence, $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0.$

THEOREM 7 *If the sums of the first $n, 2n, 3n$ terms of an AP are S_1, S_2 and S_3 respectively, prove that $S_3=3(S_2-S_1)$.*

PROOF Let *a* be the first term and *d* be the common difference of the given AP.

Then, $S_1=\frac{n}{2}\cdot\{2a+(n-1)d\}\ \Rightarrow\ S_1=na+\frac{1}{2}n(n-1)d,$ … (i)

$$S_2=\frac{2n}{2}\cdot\{2a+(2n-1)d\}\ \Rightarrow\ S_2=2na+n(2n-1)d \quad \ldots \text{(ii)}$$

and $S_3=\frac{3n}{2}\cdot\{2a+(3n-1)d\}\ \Rightarrow\ S_3=3na+\frac{3}{2}n(3n-1)d.$ … (iii)

$$\therefore\quad (S_2-S_1)=na+nd\left\{(2n-1)-\frac{1}{2}(n-1)\right\}$$
$$=na+\frac{n}{2}\cdot(3n-1)d=\frac{n}{2}\cdot\{2a+(3n-1)d\}.$$

$$\therefore\quad 3(S_2-S_1)=\frac{3n}{2}\cdot\{2a+(3n-1)d\}=S_3.$$

Hence, $S_3=3(S_2-S_1)$.

THEOREM 8 *If $S_1, S_2, S_3, \ldots, S_m$ be the sums of n terms of m arithmetic progressions whose first terms are $1, 2, 3, \ldots, m$ respectively and the common differences are $1, 3, 5, \ldots, (2m-1)$ respectively then show that*

$$S_1+S_2+S_3+\ldots+S_m=\frac{1}{2}mn(mn+1).$$

PROOF Clearly, we have

$$S_1=\frac{n}{2}\times\{2\times 1+(n-1)\times 1\} \qquad [\because\ a=1, d=1]$$
$$S_2=\frac{n}{2}\times\{2\times 2+(n-1)\times 3\} \qquad [\because\ a=2, d=3]$$
$$S_3=\frac{n}{2}\times\{2\times 3+(n-1)\times 5\} \qquad [\because\ a=3, d=5]$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$S_m=\frac{n}{2}\times\{2\times m+(n-1)(2m-1)\} \qquad [\because\ a=m, d=(2m-1)]$$

Adding columnwise, we get

$$S_1+S_2+S_3+\ldots+S_m=\frac{n}{2}\times[2\times(1+2+3+\ldots+m)]$$
$$+\frac{n(n-1)}{2}\times[1+3+5+\ldots+(2m-1)]$$

$$= \left\{n \times \frac{m}{2}(1+m)\right\} + \frac{n(n-1)}{2} \times \frac{m}{2} \times [1+(2m-1)]$$
$$= \left\{\frac{n}{2} \times m(m+1)\right\} + \left\{\frac{n}{2} \times m^2(n-1)\right\}$$
$$= \frac{mn}{2} \times [(m+1) + m(n-1)] = \frac{1}{2}mn(mn+1).$$

Hence, $S_1 + S_2 + S_3 + \ldots + S_m = \frac{1}{2}mn(mn+1)$.

THEOREM 9 *If the sum of m terms of an AP is equal to the sum of either the next n terms or the next p terms then prove that*

$$(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right).$$

PROOF Let us consider an AP with first term a and common difference d.

Then, $(T_1 + T_2 + T_3 + \ldots + T_m) = (T_{m+1} + T_{m+2} + T_{m+3} + \ldots + T_{m+n})$

$\Rightarrow \quad 2(T_1 + T_2 + T_3 + \ldots + T_m) = (T_1 + T_2 + \ldots + T_m + T_{m+1} + \ldots + T_{m+n})$

[adding $(T_1 + T_2 + T_3 + \ldots + T_m)$ on both sides]

$\Rightarrow \quad 2 \times \frac{m}{2} \times [2a + (m-1)d] = \frac{(m+n)}{2} \times [2a + (m+n-1)d]$

$\Rightarrow \quad mx = \frac{(m+n)}{2} \times (x + nd)$, where $[2a + (m-1)d] = x$

$\Rightarrow \quad 2mx = (m+n)x + (m+n)nd$

$\Rightarrow \quad (m-n)x = (m+n)nd.$ … (i)

Similarly, $(m-p)x = (m+p)pd.$ … (ii)

On dividing (i) by (ii), we get

$$\frac{(m-n)}{(m-p)} = \frac{(m+n)n}{(m+p)p}$$

$\Rightarrow \quad (m+n)(m-p)n = (m+p)(m-n)p$ [by cross multiplication]

$\Rightarrow \quad (m+n)\left(\frac{1}{p} - \frac{1}{m}\right) = (m+p)\left(\frac{1}{n} - \frac{1}{m}\right)$ [on dividing both sides by mnp]

$\Rightarrow \quad (m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right).$

Hence, $(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right).$

THEOREM 10 *The first term of an AP is a and the sum of its first p terms is zero. Then, show that the sum of its next q terms is* $\frac{-a(p+q)q}{(p-1)}$.

PROOF Let d be the common difference of the given AP. Then,

$$S_p = 0 \Rightarrow \frac{p}{2}[2a + (p-1)d] = 0$$

$$\Rightarrow 2a + (p-1)d = 0 \Rightarrow d = \frac{2a}{(1-p)}. \quad \ldots \text{(i)} \quad [\because p \neq 0]$$

Let S be the sum of next q terms. Then,

$$S = T_{p+1} + T_{p+2} + \ldots + T_{p+q}$$

$$= (T_1 + T_2 + \ldots + T_p + T_{p+1} + T_{p+2} + \ldots + T_{p+q}) - (T_1 + T_2 + \ldots + T_p)$$

$$= S_{p+q} - S_p = S_{p+q} - 0 = S_{p+q} \qquad [\because\ S_p = 0]$$

$$= \frac{(p+q)}{2} \cdot [2a + (p+q-1)d]$$

$$= \frac{(p+q)}{2} \cdot \left[2a + (p+q-1) \times \frac{2a}{(1-p)}\right] \qquad \text{[using (i)]}$$

$$= (p+q)a \cdot \left[1 + \frac{(p+q-1)}{(1-p)}\right] = \frac{-a(p+q)q}{(p-1)}.$$

Hence, the sum of next q terms is $\frac{-a(p+q)q}{(p-1)}$.

THEOREM 11 *If the sum of n terms of an AP is $(cn + dn^2)$, where c and d are constants then show that the common difference of the AP is $2d$.*

PROOF Let $S_n = cn + dn^2$ … (i), where c and d are constants.

Replacing n by $(n-1)$ in (i), we get

$$S_{n-1} = c(n-1) + d(n-1)^2$$

$$\Rightarrow \quad S_{n-1} = dn^2 + n(c-2d) + (d-c). \qquad \ldots \text{(ii)}$$

On subtracting (ii) from (i), we get

$$(S_n - S_{n-1}) = 2dn + (c-d)$$

$$\Rightarrow \quad T_n = 2dn + (c-d). \qquad \ldots \text{(iii)}$$

Replacing n by $(n-1)$ in (iii), we get

$$T_{n-1} = 2d(n-1) + (c-d). \qquad \ldots \text{(iv)}$$

On subtracting (iv) from (iii), we get

$(T_n - T_{n-1}) = 2d$, which is constant.

Hence, the common difference of the given AP is $2d$.

EXERCISE 11B

1. Find the sum of 23 terms of the AP $17, 12, 7, 2, -3, \ldots$.
2. Find the sum of 16 terms of the AP $6, 5\frac{1}{3}, 4\frac{2}{3}, 4, \ldots$.
3. Find the sum of 25 terms of the AP $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, \ldots$.
4. Find the sum of 100 terms of the AP $0.6, 0.61, 0.62, 0.63, \ldots$.
5. Find the sum of 20 terms of the AP $(x+y), (x-y), (x-3y), \ldots$.
6. Find the sum of n terms of the AP $\frac{x-y}{x+y}, \frac{3x-2y}{x+y}, \frac{5x-3y}{x+y}, \ldots$.
7. Find the sum of the series $2 + 5 + 8 + 11 + \ldots + 191$.
8. Find the sum of the series $101 + 99 + 97 + 95 + \ldots + 43$.

9. Find the value of x such that $1+4+7+10+\ldots+x=715$.
10. Find the value of x such that $25+22+19+16+\ldots+x=112$.
11. Find the rth term of an AP, the sum of whose first n terms is $(3n^2+2n)$.
12. Find the sum of n terms of an AP whose rth term is $(5r+1)$.
13. If the sum of a certain number of terms of the AP 27, 24, 21, 18, ... is –30, find the last term.
14. How many terms of the AP 26, 21, 16, 11, ... are needed to give the sum 11?
15. How many terms of the AP 18, 16, 14, 12, ... are needed to give the sum 78? Explain the double answer.
16. How many terms of the AP $20, 19\frac{1}{3}, 18\frac{2}{3}, \cdots$ must be taken to make the sum 300? Explain the double answer.
17. The sums of n terms of two arithmetic progressions are in the ratio $(7n-5):(5n+17)$. Show that their 6th terms are equal.
18. If the ratio between the sums of n terms of two arithmetic progressions is $(7n+1):(4n+27)$, find the ratio of their 11th terms.
19. Find the sum of all odd integers from 1 to 201.
20. Find the sum of all even integers between 101 and 199.
21. Find the sum of all integers between 101 and 500, which are divisible by 9.
22. Find the sum of all integers between 100 and 600, each of which when divided by 5 leaves 2 as remainder.
23. The sum of first 7 terms of an AP is 10 and that of next 7 terms is 17. Find the AP.
24. If the sum of n terms of an AP is $(3n^2+5n)$ and its mth term is 164, find the value of m.
25. Find the sum of all natural numbers from 1 and 100 which are divisible by 4 or 5.
26. If the sum of n terms of an AP is $\left\{nP+\frac{1}{2}n(n-1)Q\right\}$, where P and Q are constants then find the common difference.
27. If $S_m=m^2p$ and $S_n=n^2p$, where $m\neq n$ in an AP then prove that $S_p=p^3$.
28. A carpenter was hired to build 192 window frames. The first day he made 5 frames and each day, thereafter he made 2 more frames than he made the day before. How many days did he take to finish the job?

ANSWERS (EXERCISE 11B)

1. –874 **2.** 16 **3.** $325\sqrt{2}$ **4.** 109.5 **5.** $20(x-18y)$

6. $\frac{n}{2(x+y)}\cdot\{n(2x-y)-y\}$ **7.** 6176 **8.** 2160 **9.** $x=64$

10. $x=7$ **11.** $T_r=(6r-1)$ **12.** $S_n=\frac{1}{2}n(5n+7)$ **13.** –30

14. 11 **15.** 6 and 13, sum of all terms from 7th to 13th is zero

16. 25 and 36, sum of all terms from 26th to 36th is zero **18.** 4 : 3

19. 10201 **20.** 7350 **21.** 13266 **22.** 34950

23. $1+1\frac{1}{7}+1\frac{2}{7}+1\frac{3}{7}+\ldots$ **24.** $m=27$ **25.** 2050 **26.** Q

28. 12 days

HINTS TO SOME SELECTED QUESTIONS

5. $a=(x+y)$ and $d=(x-y)-(x+y)=-2y$.

$S_{20}=\frac{20}{2}\times[2a+(20-1)d]=10\times[2(x+y)+19\times(-2y)]=20(x-18y)$.

6. Required sum $=\frac{1}{(x+y)}\cdot\{(x-y)+(3x-2y)+(5x-3y)+\ldots \text{ to } n \text{ terms}]$

$=\frac{1}{(x+y)}\cdot\frac{n}{2}\cdot[2(x-y)+(n-1)(2x-y)]$ $[\because\ a=(x-y) \text{ and } d=(2x-y)]$

$=\frac{n}{2(x+y)}\cdot\{n(2x-y)-y\}$.

9. Here, $a=1$ and $d=3$. Let it contain n terms. Then,

$\frac{n}{2}\cdot[2a+(n-1)d]=715\ \Rightarrow\ \frac{n}{2}\cdot[2\times1+(n-1)\times3]=715$

$\Rightarrow\quad n(3n-1)=1430\ \Rightarrow\ 3n^2-n-1430=0\ \Rightarrow\ 3n^2-66n+65n-1430=0$

$\Rightarrow\quad 3n(n-22)+65(n-22)=0\ \Rightarrow\ (n-22)(3n+65)=0\ \Rightarrow\ n=22.$

$\therefore\quad x=T_{22}=(a+21d)=(1+21\times3)=64.$

10. $3n^2-21n-32n+224=0.$

11. $S_n=3n^2+2n\ \Rightarrow\ S_{n-1}=3(n-1)^2+2(n-1)\ \Rightarrow\ S_{n-1}=3n^2-4n+1.$

$\therefore\quad T_n=(S_n-S_{n-1})=(3n^2+2n)-(3n^2-4n+1)=(6n-1)\ \Rightarrow\ T_r=(6r-1).$

13. Here, $a=27$, $d=(24-27)=-3$ and $S_n=-30$. Then,

$\frac{n}{2}\cdot[2a+(n-1)d]=-30\ \Rightarrow\ \frac{n}{2}\cdot[54+(n-1)\times(-3)]=-30$

$\therefore\quad 3n^2-57n-60=0\ \Rightarrow\ n^2-19n-20=0\ \Rightarrow\ n=20.$

Now, $S_n=-30\ \Rightarrow\ \frac{n}{2}(a+l)=-30\ \Rightarrow\ \frac{20}{2}(27+l)=-30\ \Rightarrow\ l=-30.$

16. Here, $a=20$ and $d=\left(\frac{58}{3}-20\right)=\frac{-2}{3}$. Let $S_n=300$. Then,

$\frac{n}{2}\times[2a+(n-1)d]=300\ \Rightarrow\ \frac{n}{2}\times\left[40+(n-1)\times\left(\frac{-2}{3}\right)\right]=300$

$\therefore\quad \frac{n}{2}\cdot\frac{(122-2n)}{3}=300\ \Rightarrow\ n(61-n)=900\ \Rightarrow\ n^2-61n+900=0$

$\Rightarrow\quad n^2-36n-25n+900=0\ \Rightarrow\ n(n-36)-25(n-36)=0\ \Rightarrow\ (n-25)(n-36)=0.$

$\therefore\quad S_{25}=300$ and $S_{36}=300.$

Sum of all terms from 26th to 36th is zero.

17. $\frac{\frac{n}{2}\cdot[2a_1+(n-1)d_1]}{\frac{n}{2}\cdot[2a_2+(n-1)d_2]}=\frac{7n-5}{5n+17}\ \Rightarrow\ \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2}=\frac{7n-5}{5n+17}$

$$\Rightarrow \quad \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_2+\frac{(n-1)}{2}d_2}=\frac{7n-5}{5n+17} \Rightarrow \frac{a_1+5d_1}{a_2+5d_2}=\frac{7\times 11-5}{5\times 11+17} \qquad \left[\frac{n-1}{2}=5 \Rightarrow n=11\right]$$

$$=\frac{72}{72}=1.$$

Hence, their 6th terms are equal.

18. Proceeding as above, we get

$$\frac{a_1+\frac{1}{2}(n-1)d_1}{a_2+\frac{1}{2}(n-1)d_2}=\frac{7n+1}{4n+27} \Rightarrow \frac{a_1+10d_1}{a_2+10d_2}=\frac{7\times 21+1}{4\times 21+27} \qquad \left[\frac{n-1}{2}=10 \Rightarrow n=21\right]$$

$$=\frac{148}{111}=\frac{4}{3}.$$

19. Required sum $=1+3+5+7+\ldots+201.$

$T_n=201 \Rightarrow 1+(n-1)\times 2=201 \Rightarrow n=101.$

$\therefore \quad S_n=\frac{101}{2}\times(1+201)=(101\times 101)=10201.$

20. Required sum $=102+104+106+\ldots+198.$

21. Required sum $=108+117+126+\ldots+495.$

22. Required sum $=102+107+112+\ldots+597.$

23. $S_7=10$ and $S_{14}=(17+10)=27.$

$\therefore \quad \frac{7}{2}\times[2a+6d]=10 \Rightarrow 7a+21d=10 \qquad \ldots$ (i)

and $\frac{14}{2}\times[2a+13d]=27 \Rightarrow 14a+91d=27. \qquad \ldots$ (ii)

Solve for a and d.

24. $T_n=(S_n-S_{n-1})=(3n^2+5n)-(3n^2-n-2)=(6n+2).$

$T_m=164 \Rightarrow 6m+2=164 \Rightarrow 6m=162 \Rightarrow m=27.$

25. Required numbers = (numbers divisible by 4) + (numbers divisible by 5)

− (numbers divisible by 20)

$=(4+8+12+\ldots+100)+(5+10+15+\ldots+100)$

$-(20+40+60+\ldots+100)$

$=\frac{25}{2}\times(4+100)+\frac{20}{2}\times(5+100)-\frac{5}{2}\times(20+100)$

$=(1300+1050-300)=2050.$

26. $S_n=nP+\frac{1}{2}n(n-1)Q \Rightarrow S_{n-1}=(n-1)P+\frac{1}{2}(n-1)(n-2)Q.$

$\therefore \quad T_n=(S_n-S_{n-1})=P+(n-1)Q \Rightarrow T_{n-1}=P+(n-2)Q.$

$\therefore \quad d=(T_n-T_{n-1})=Q.$

27. $S_m=m^2p$ and $S_{m-1}=(m-1)^2p \Rightarrow S_m-S_{m-1}=\{m^2-(m-1)^2\}p=(2m-1)p.$

$T_m=(2m-1)p \Rightarrow T_1=p, T_2=3p, T_3=5p,\ldots \Rightarrow a=p$ and $d=2p.$

$S_p=\frac{p}{2}\times[2p+(p-1)\times 2p]=\left(\frac{p}{2}\times 2p^2\right)=p^3.$

28. We have $5+7+9+\ldots=192$

$\frac{n}{2}\times[10+(n-1)\times 2]=192 \Rightarrow n(n+4)=192 \Rightarrow n^2+4n-192=0$

$\Rightarrow (n+16)(n-12)=0 \Rightarrow n=12.$

EXERCISE 11C

1. The interior angles of a polygon are in AP. The smallest angle is 52° and the common difference is 8°. Find the number of sides of the polygon.
2. A circle is completely divided into n sectors in such a way that the angles of the sectors are in AP. If the smallest of these angles is 8° and the largest is 72°, calculate n and the angle in the fifth sector.
3. There are 30 trees at equal distances of 5 metres in a line with a well, the distance of the well from the nearest tree being 10 metres. A gardner waters all the trees separately starting from the well and he returns to the well after watering each tree to get water for the next. Find the total distance the gardner will cover in order to water all the trees.
4. Two cars start together from the same place in same direction. The first goes with uniform speed of 60 km/ hr. The second goes at a speed of 48 km/ hr in the first hour and increases the speed by 1 km each succeeding hour. After how many hours will the second car overtake the first car if both cars go non-stop?
5. Arun buys a scooter for ` 44000. He pays ` 8000 in cash and agrees to pay the balance in annual instalments of ` 4000 each plus 10% interest on the unpaid amount. How much did he pay for it?
6. A man accepts a position with an initial salary of ` 26000 per month. It is understood that he will receive an automatic increase of ` 250 in the very next month and each month thereafter.

   Find his (i) salary for the 10th month, (ii) total earnings during the first year.
7. A man saved ` 660000 in 20 years. In each succeeding year after the first year he saved ` 2000 more than what he saved in the previous year. How much did he save in the first year?
8. 150 workers were engaged to finish a piece of work in a certain number of days. Four workers dropped the second day, four more workers dropped the third day, and so on. It takes 8 more days to finish the work now. Find the number of days in which the work was completed.
9. A man saves ` 4000 during first year, ` 5000 during second year and in this way he increases his savings by ` 1000 every year. Find in what time his savings will be ` 85000.
10. A man arranges to pay off a debt of ` 36000 by 40 annual instalments which form an AP. When 30 of the instalments are paid, he dies, leaving one-third of the debt unpaid. Find the value of first instalment.
11. A manufacturer of TV sets produced 6000 units in the third year and 7000 units in the seventh year. Assuming that the production increases uniformly by a fixed number every year, find the production (i) in the first year, (ii) in the 10th year, (iii) in 7 years.

12. A farmer buys a used tractor for ` 180000. He pays ` 90000 in cash and agrees to pay the balance in annual instalments of ` 9000 plus 12% interest on the unpaid amount. How much did the tractor cost him?

ANSWERS (EXERCISE 11C)

1. 3 **2.** $n = 9$, 5th angle = $40°$ **3.** 4795 m **4.** 25 hours **5.** ` 62000
6. (i) ` 28250 (ii) ` 328500 **7.** ` 14000 **8.** 25 days **9.** 10 years
10. ` 510 **11.** (i) 5500 (ii) 7750 (iii) 43750 **12.** ` 239400

HINTS TO SOME SELECTED QUESTIONS

1. Let the number of sides of the polygon be *n*. Then,

sum of its interior angles $= (2n - 4) \times 90° = (180n - 360)°$.

$\therefore$ $52 + 60 + 68 + \ldots$ to *n* terms $= (180n - 360)$

$\Rightarrow$ $\frac{n}{2} \times [2 \times 52 + (n-1) \times 8] = 180n - 360$

$\Rightarrow$ $n^2 - 33n + 90 = 0 \Rightarrow (n-3)(n-30) = 0 \Rightarrow n = 3$ or $n = 30$.

When $n = 30$, then last angle $= [52 + 29 \times 8]° = 284°$, which is a reflex angle. So, $n \neq 30$.

Hence, $n = 3$.

2. Let the number of sectors be *n*. Then,

$$\frac{n}{2}(a + l) = 360 \Rightarrow n(8 + 72) = 720 \Rightarrow n = \frac{720}{80} = 9.$$

Now, $a = 8$, $T_n = 72$ and $n = 9$.

$\therefore$ $8 + 8d = 72 \Rightarrow 8d = 64 \Rightarrow d = 8$.

So, the AP is 8, 16, 24, 32, ... So, 5th angle $= (a + 4d) = 40°$.

3. Total distance covered in metres

$= 2 \times \{10 + 15 + 20 + \ldots$ up to 29 terms$\}$ + 30th term

$= 2 \times \frac{29}{2} \times \{20 + 28 \times 5\} + (10 + 29 \times 5)$

$= (4640 + 155) = 4795$ m.

4. Suppose the second car overtakes the first car in *n* hours.

Distance covered by the first car in *n* hours $= (60n)$ km.

Distance covered by the second car in *n* hours

$= [48 + 49 + 50 + \ldots$ to *n* terms$]$

$= \frac{n}{2} \times [2 \times 48 + (n-1) \times 1] = \frac{n}{2}(95 + n)$.

$\therefore$ $\frac{n}{2}(95 + n) = 60n \Rightarrow 95 + n = 120 \Rightarrow n = 25$ hours.

5. Total cost = ` 44000. Paid in cash = ` 8000. Balance = ` 36000.

Amount of each instalment = ` 4000.

Number of instalments $= 36000 \div 4000 = 9$.

1st instalment = ` $\left(4000 + 36000 \times \frac{10}{100} \times 1\right) =$ ` 7600.

2nd instalment = ` $\left(4000 + 32000 \times \frac{10}{100} \times 1\right) =$ ` 7200.

3rd instalment = ` $\left(4000 + 28000 \times \frac{10}{100} \times 1\right) =$ ` 6800, and so on.

Total payment = ` $\{8000 + (7600 + 7200 + 6800 + \ldots \text{ to 9 terms})\}$

$= ` \left\{8000 + \frac{9}{2}(15200 + 8 \times (-400))\right\} = ` \ 62000.$

6. The amounts form an AP with $a = ` \ 26000$ and $d = ` \ 250.$

(i) $T_{10} = (a + 9d) = ` \ (26000 + 9 \times 250) = ` \ 28250.$

(ii) Total earning during 1 year $= ` \left\{\frac{12}{2} \times (2 \times 26000 + 11 \times 250)\right\} = ` \ 328500.$

7. Suppose that the man saved ` x in first year. Then,

$x + (x + 2000) + (x + 4000) + \ldots$ to 20 terms = 660000.

This is an AP in which $a = x, d = 2000$ and $n = 20.$

$\therefore \quad S_{20} = 660000 \Rightarrow \frac{20}{2} \times (2x + 19 \times 2000) = 660000 \Rightarrow x = 14000.$

8. Suppose that the work was completed in n days.

Total number of workers engaged for 1 day each

$= 150 + 146 + 142 + \ldots$ to n terms

$= \frac{n}{2} \times [2 \times 150 + (n - 1) \times (-4)] = n(152 - 2n).$

If 150 workers are engaged each day, time taken $= (n - 8)$ days.

Number of workers engaged for 1 day $= 150(n - 8).$

$\therefore \quad n(152 - 2n) = 150(n - 8) \Rightarrow n^2 - n - 600 = 0$

$\Rightarrow (n - 25)(n + 24) = 0$

$\Rightarrow n = 25 \qquad [\because \ n \neq -24].$

9. Let the total savings be ` 85000 in n years. Then,

$\frac{n}{2} \times [2 \times 4000 + (n - 1) \times 1000] = 85000$

$\Rightarrow \quad n^2 + 7n - 170 = 0 \Rightarrow (n + 17)(n - 10) = 0 \Rightarrow n = 10$ years.

10. Let the first instalment be ` $a$ and let it increase by ` d each year.

Then, $S_{30} = \left(\frac{2}{3} \times 36000\right) = 24000$ and $S_{40} = 36000$

$\Rightarrow \quad \frac{30}{2} \times (2a + 29d) = 24000$ and $\frac{40}{2} \times (2a + 39d) = 36000$

$\Rightarrow \quad 2a + 29d = 1600 \qquad \ldots$ (i) and $2a + 39d = 1800. \qquad \ldots$ (ii)

On solving (i) and (ii), we get $a = 510$ and $d = 20.$

$\therefore$ first instalment = ` 510.

11. Let the production in first year be a, increasing uniformly at the rate of d per year. Then,

$a + 2d = 6000$ and $a + 6d = 7000.$

On solving, we get $a = 5500$ and $d = 250.$

(i) Production in the 1st year $= a = 5500.$

(ii) Production in the 10th year $= (a + 9d) = (5500 + 9 \times 250).$

(iii) Production in 7 years $= \frac{7}{2} \times [2a + (7 - 1)d] = \frac{7}{2} \times [2 \times 5500 + 6 \times 250].$

12. Total cost = ` 180000. Paid in cash = ` 90000. Balance = ` 90000.

Amount of each instalment = ` 9000.

Number of instalments $= \frac{90000}{9000} = 10.$

1st instalment = ` $\left(9000 + 90000 \times \frac{12}{100} \times 1\right)$ = ` 19800.

2nd instalment = ` $\left(9000 + 81000 \times \frac{12}{100} \times 1\right)$ = ` 18720.

3rd instalment = ` $\left(9000 + 72000 \times \frac{12}{100} \times 1\right)$ = ` 17640, and so on.

Total payment = ` [90000 + (19800 + 18720 + 17640 + ... up to 10 terms]

$= ` \left[90000 + \frac{10}{2} \cdot \{39600 + 9 \times (-1080)\}\right]$

$= ` (90000 + 149400) = ` 239400.$

ARITHMETIC MEAN

ARITHMETIC MEAN

If a, A, b are in AP then we say that A is the arithmetic mean (AM) between a and b.

INSERTION OF A SINGLE ARITHMETIC MEAN BETWEEN *a* AND *b*

Let *a* and *b* be two given numbers and let *A* be the arithmetic mean between *a* and *b*. Then,

a, A, b are in AP

$\Rightarrow \quad A - a = b - A$

$\Rightarrow \quad 2A = a + b \Rightarrow A = \frac{a+b}{2}.$

Hence, the arithmetic mean between *a* and *b* is $\frac{a+b}{2}$.

EXAMPLE 1 *Find the arithmetic mean between*

(i) 14 *and* –6, *(ii)* $(a - b)$ *and* $(a + b)$.

SOLUTION (i) Arithmetic mean between 14 and –6

$$= \frac{14 + (-6)}{2} = \frac{8}{2} = 4.$$

(ii) Arithmetic mean between $(a - b)$ and $(a + b)$

$$= \frac{(a-b) + (a+b)}{2} = \frac{2a}{2} = a.$$

EXAMPLE 2 *If* $\left(\frac{a^n + b^n}{a^{n-1} + b^{n-1}}\right)$ *is the AM between a and b then find the value of n.*

SOLUTION We know that the AM between *a* and *b* is $\frac{(a+b)}{2}$.

$\therefore \quad \left(\frac{a^n + b^n}{a^{n-1} + b^{n-1}}\right)$ is the AM between *a* and *b*

$\Rightarrow \quad \frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \frac{a+b}{2}$

$\Rightarrow \quad 2a^n + 2b^n = a^n + a^{n-1}b + b^{n-1}a + b^n$

$\Rightarrow \quad a^n + b^n = a^{n-1}b + b^{n-1}a$

$\Rightarrow \quad a^n - a^{n-1}b = b^{n-1}a - b^n$

$\Rightarrow \quad a^{n-1}(a-b) = b^{n-1}(a-b)$

$\Rightarrow \quad a^{n-1} = b^{n-1}$ [on dividing both sides by $(a-b)$]

$\Rightarrow \quad \left(\frac{a}{b}\right)^{n-1} = 1 = \left(\frac{a}{b}\right)^0 \Rightarrow n-1 = 0 \Rightarrow n = 1.$

Hence, the required value of *n* is 1.

n ARITHMETIC MEANS BETWEEN *a* AND *b*

If $a, A_1, A_2, \ldots, A_n, b$ are in AP then we say that $A_1, A_2, \ldots, A_n$ are the n arithmetic means between a and b.

INSERTION OF *n* ARITHMETIC MEANS BETWEEN *a* AND *b*

Let *a* and *b* be two given numbers and let $A_1, A_2, \ldots, A_n$ be the *n* arithmetic means between *a* and *b*. Then,

$a, A_1, A_2, \ldots, A_n, b$ are in AP.

In this AP, we have first term = a, last term = b, number of terms = $(n+2)$.

Let the common difference be *d*. Then,

$b = T_{n+2} \Rightarrow b = a + (n+2-1)d$

$\Rightarrow d = \frac{b-a}{n+1}.$ ... (i)

$\therefore \quad A_1 = (a+d), A_2 = (a+2d), A_3 = (a+3d), \ldots, A_n = (a+nd)$

$\Rightarrow \quad A_1 = \left\{a + \frac{(b-a)}{(n+1)}\right\}, A_2 = \left\{a + \frac{2(b-a)}{(n+1)}\right\}, \ldots, A_n = \left\{a + \frac{n(b-a)}{(n+1)}\right\}.$

These are the required *n* arithmetic means between *a* and *b*.

EXAMPLE 3 *Insert six arithmetic means between* 15 *and* –13.

SOLUTION Let $A_1, A_2, A_3, A_4, A_5, A_6$ be the six arithmetic means between 15 and –13. Then,

$15, A_1, A_2, A_3, A_4, A_5, A_6, -13$ are in AP.

Now, $d = \frac{(-13-15)}{(6+1)} = \frac{-28}{7} = -4 \qquad \left[\because d = \frac{(b-a)}{(n+1)} = \frac{(-13-15)}{(6+1)}\right].$

$\therefore \quad A_1 = (15+d) = (15-4) = 11, A_2 = (15+2d) = (15-8) = 7,$

$A_3 = (15+3d) = (15-12) = 3, A_4 = (15+4d) = (15-16) = -1,$

$A_5 = (15+5d) = (15-20) = -5, A_6 = (15+6d) = (15-24) = -9.$

Hence, the required six AMs between 15 and –13 are

11, 7, 3, –1, –5 and –9.

EXAMPLE 4 *If m arithmetic means are inserted between* 1 *and* 31 *so that the ratio of the 7th and $(m-1)$th means is* 5 : 9 *then find the value of m.*

SOLUTION Let $A_1, A_2, \ldots, A_m$ be the *m* arithmetic means between 1 and 31.

Then, $1, A_1, A_2, \ldots, A_m, 31$ are in AP.

$\therefore \quad d = \frac{(31-1)}{(m+1)} = \frac{30}{m+1} \qquad \left[\because d = \frac{(b-a)}{(m+1)}\right].$

7th mean $= A_7 = T_8 = (a + 7d) = \left(1 + \frac{7 \times 30}{m+1}\right) = \frac{(m+211)}{(m+1)}$.

$(m-1)$th mean $= A_{m-1} = T_m = a + (m-1)d = \left\{1 + \frac{(m-1) \times 30}{(m+1)}\right\}$

$= \frac{(31m-29)}{(m+1)}$.

Now, $\frac{A_7}{A_{m-1}} = \frac{5}{9} \Rightarrow \frac{(m+211)}{(m+1)} \times \frac{(m+1)}{(31m-29)} = \frac{5}{9}$

$\Rightarrow \frac{(m+211)}{(31m-29)} = \frac{5}{9} \Rightarrow 9m + 1899 = 155m - 145$

$\Rightarrow 146m = 2044 \Rightarrow m = \frac{2044}{146} = 14.$

Hence, $m = 14$.

EXAMPLE 5 *If x, y, z are in AP and A_1 is the AM between x and y while A_2 is the AM between y and z then prove that the AM between A_1 and A_2 is y.*

SOLUTION Since *x, y, z* are in AP, we have $x + z = 2y$. ... (i)

Since A_1 is the AM between *x* and *y*, we have $A_1 = \frac{x+y}{2}$. ... (ii)

Since A_2 is the AM between *y* and *z*, we have $A_2 = \frac{y+z}{2}$. ... (iii)

$\therefore$ AM between A_1 and $A_2 = \frac{1}{2}(A_1 + A_2) = \frac{1}{2}\left\{\frac{x+y}{2} + \frac{y+z}{2}\right\}$

$= \frac{x+2y+z}{4} = \frac{4y}{4} = y$ [using (i)].

Hence, the AM between A_1 and A_2 is *y*.

EXAMPLE 6 *If the AM between pth and qth terms of an AP be equal to the AM between its rth and sth terms then prove that $p + q = r + s$.*

SOLUTION Let *a* be the first term and *d* be the common difference of the given AP. Then,

$\frac{1}{2}(T_p + T_q) = \frac{1}{2}(T_r + T_s)$

$\Rightarrow \quad T_p + T_q = T_r + T_s$

$\Rightarrow \quad \{a + (p-1)d\} + \{a + (q-1)d\} = \{a + (r-1)d\} + \{a + (s-1)d\}$

$\Rightarrow \quad 2a + (p+q-2)d = 2a + (r+s-2)d$

$\Rightarrow \quad (p+q-2)d = (r+s-2)d$

$\Rightarrow \quad p + q - 2 = r + s - 2 \Rightarrow p + q = r + s.$

Hence, $p + q = r + s$.

EXAMPLE 7 *The n arithmetic means between 20 and 80 are such that the first mean : last mean = 1 : 3. Find the value of n.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n AMs between 20 and 80.

Then, $20, A_1, A_2, \ldots, A_n, 80$ are in AP.

Let their common difference be d. Then,

$$T_{n+2} = 80 \Rightarrow 20 + (n+2-1)d = 80 \Rightarrow d = \frac{60}{(n+1)}. \qquad \ldots \text{(i)}$$

First mean, $A_1 = T_2 = (20 + d) = \left(20 + \frac{60}{n+1}\right)$ [using (i)]

$$= \frac{(20n+80)}{(n+1)}.$$

Last mean, $A_n = T_{n+1} = [20 + (n+1-1)d]$

$$= \left\{20 + \frac{n \times 60}{(n+1)}\right\} \quad \text{[using (i)]}$$

$$= \frac{(80n+20)}{(n+1)}.$$

$$\therefore \quad \frac{A_1}{A_n} = \frac{1}{3} \Rightarrow \frac{(20n+80)}{(n+1)} \times \frac{(n+1)}{(80n+20)} = \frac{1}{3}$$

$$\Rightarrow 3(20n+80) = (80n+20) \Rightarrow 20n = 220 \Rightarrow n = 11.$$

Hence, $n = 11$.

EXAMPLE 8 *If n arithmetic means are inserted between two numbers, prove that the sum of the means equidistant from the beginning and the end is constant.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n AMs between a and b.

Then, $a, A_1, A_2, \ldots, A_n, b$ are in AP.

$\therefore$ mth mean from the beginning + mth mean from the end

= $(m+1)$th term from the beginning

+ $(m+1)$th term from the end

$= \{a + (m+1-1)d\} + \{b - (m+1-1)d\} = (a+b) =$ constant.

Hence, the sum of the means equidistant from the beginning and the end is constant.

EXAMPLE 9 *Show that the sum of n arithmetic means between two numbers is n times the single arithmetic mean between them.*

SOLUTION Let a and b be two given numbers and let A be the single AM between them. Then, $A = \frac{a+b}{2}$.

Let $A_1, A_2, \ldots, A_n$ be the n arithmetic means between a and b.

Then $a, A_1, A_2, \ldots, A_n, b$ are in AP.

Now, $(A_1 + A_2 + \ldots + A_n) = \frac{n}{2}(A_1 + A_n)$ $\left[S_n = \frac{n}{2}(a+l)\right]$

$= \frac{n}{2}(a+b)$ [$\because$ $a, A_1, A_2, \ldots, A_n, b$ is an AP

$\Rightarrow a + b = A_1 + A_n$]

$= n \times$ (AM between a and b).

Hence, the sum of n arithmetic means between a and b is n times the AM between a and b.

EXERCISE 11D

1. Find the arithmetic mean between
   (i) 9 and 19, (ii) 15 and –7, (iii) –16 and –8.
2. Insert four arithmetic means between 4 and 29.
3. Insert three arithmetic means between 23 and 7.
4. Insert six arithmetic means between 11 and –10.
5. There are n arithmetic means between 9 and 27. If the ratio of the last mean to the first mean is 2 : 1, find the value of n.
6. Insert arithmetic means between 16 and 65 such that the 5th AM is 51. Find the number of arithmetic means.
7. Insert five numbers between 11 and 29 such that the resulting sequence is an AP.
8. Prove that the sum of m arithmetic means between the two numbers is to the sum of n arithmetic means between them is $m : n$.

ANSWERS (EXERCISE 11D)

1. (i) 14 (ii) 4 (iii) –12 **2.** 9, 14, 19, 24 **3.** 19, 15, 11 **4.** 8, 5, 2, –1, –4, –7
5. $n = 5$ **6.** six **7.** 14, 17, 20, 23, 26

HINTS TO SOME SELECTED QUESTIONS

5. Let $A_1, A_2, \ldots, A_n$ be the n arithmetic means between 9 and 27.

Then, $9, A_1, A_2, \ldots, A_n, 27$ are in AP.

$\therefore \quad 9 + (n+2-1)d = 27 \Rightarrow d = \dfrac{18}{n+1}.$

$$\frac{A_n}{A_1} = \frac{2}{1} \Rightarrow \frac{T_{n+1}}{T_2} = \frac{2}{1} \Rightarrow \frac{9 + (n+1-1)d}{9 + (2-1)d} = \frac{2}{1}$$

$$\Rightarrow 9 + nd = 18 + 2d \Rightarrow d = \frac{9}{n-2}.$$

$\therefore \quad \dfrac{18}{n+1} = \dfrac{9}{n-2} \Rightarrow 18(n-2) = 9(n+1) \Rightarrow n = 5.$

6. $16, A_1, A_2, A_3, A_4, A_5, \ldots, A_n, 65$ are in AP.

Then, $T_6 = 51 \Rightarrow a + 5d = 51 \Rightarrow 16 + 5d = 51 \Rightarrow d = 7.$

$T_{n+2} = 65 \Rightarrow 16 + (n+2-1)d = 65 \Rightarrow 16 + (n+1) \times 7 = 65 \Rightarrow n = 6.$

8. Sum of m AMs between a and b

$$= m \times (\text{AMs between } a \text{ and } b) = m \times \frac{(a+b)}{2}.$$

Sum of n AMs between a and b

$$= n \times (\text{AMs between } a \text{ and } b) = n \times \frac{(a+b)}{2}.$$

$\therefore$ required ratio $= m : n.$

PROPERTIES OF AP

THEOREM 1 *If to each term of an AP a fixed nonzero number is added then the resulting progression is an AP.*

PROOF Let $a, a+d, a+2d, a+3d, \ldots$ be a given AP and let k be a fixed nonzero number. Then,

$(a+k), \{(a+d)+k\}, \{(a+2d)+k\}, \ldots$ is an AP

in which first term $= (a+k)$ and common difference $= d$.

THEOREM 2 *If each term of a given AP is multiplied or divided by a given nonzero fixed number k then the resulting progression is an AP.*

PROOF (i) Let $a, a+d, a+2d, \ldots$ be the given AP and let k be a given nonzero number. Then,

$ak, (a+d)k, (a+2d)k, \ldots$ is clearly an AP

in which the first term $= ak$ and the common difference $= dk$.

(ii) Let $a, a+d, a+2d, \ldots$ be the given AP and let k be a given nonzero number. Then,

$\frac{a}{k}, \frac{(a+d)}{k}, \frac{(a+2d)}{k}, \cdots$ is clearly an AP

in which the first term $= \frac{a}{k}$ and the common difference $= \frac{d}{k}$.

SUMMARY

If $a_1, a_2, a_3, a_4, \ldots$ are in AP and k is a nonzero number then

(i) $(a_1+k), (a_2+k), (a_3+k), (a_4+k), \ldots$ are in AP.

(ii) $(a_1-k), (a_2-k), (a_3-k), (a_4-k), \ldots$ are in AP.

(iii) $a_1k, a_2k, a_3k, a_4k, \ldots$ are in AP.

(iv) $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \frac{a_4}{k}, \cdots$ are in AP.

SOLVED EXAMPLES

EXAMPLE 1 *If a, b, c, are in AP, show that*

(i) $(b+c), (c+a)$ *and* $(a+b)$ *are in AP.*

(ii) $a^2(b+c), b^2(c+a)$ *and* $c^2(a+b)$ *are in AP.*

SOLUTION Since a, b, c are in AP, we have

$$2b = (a+c). \qquad \ldots (i)$$

(i) $(b+c), (c+a), (a+b)$ will be in AP

if $(c+a)-(b+c) = (a+b)-(c+a)$

i.e., if $a-b = b-c$

i.e., if $2b = a+c$, which is true by (i).

$\therefore$ a, b, c are in AP $\Rightarrow$ $(b+c), (c+a), (a+b)$ are in AP.

(ii) $a^2(b+c), b^2(c+a), c^2(a+b)$ will be in AP

if $b^2(c+a)-a^2(b+c)=c^2(a+b)-b^2(c+a)$

i.e., if $c(b^2-a^2)+ab(b-a)=a(c^2-b^2)+bc(c-b)$

i.e., if $(b-a)(ab+bc+ca)=(c-b)(ab+bc+ca)$

i.e., if $(b-a)=(c-b)$

i.e., if $2b=(a+c)$, which is true by (i).

$\therefore$ *a, b, c* are in AP $\Rightarrow$ $a^2(b+c), b^2(c+a), c^2(a+b)$ are in AP.

EXAMPLE 2 *If a, b, c are in AP, show that*

$$\frac{1}{(\sqrt{b}+\sqrt{c})}, \frac{1}{(\sqrt{c}+\sqrt{a})}, \frac{1}{(\sqrt{a}+\sqrt{b})} \text{ are in AP.}$$

SOLUTION Since *a, b, c* are in AP, we have

$$2b=(a+c). \qquad \ldots \text{(i)}$$

Now, $\frac{1}{(\sqrt{b}+\sqrt{c})}, \frac{1}{(\sqrt{c}+\sqrt{a})}, \frac{1}{(\sqrt{a}+\sqrt{b})}$ will be in AP

if $\frac{1}{(\sqrt{c}+\sqrt{a})}-\frac{1}{(\sqrt{b}+\sqrt{c})}=\frac{1}{(\sqrt{a}+\sqrt{b})}-\frac{1}{(\sqrt{c}+\sqrt{a})}$

i.e., if $\frac{(\sqrt{b}+\sqrt{c})-(\sqrt{c}+\sqrt{a})}{(\sqrt{c}+\sqrt{a})(\sqrt{b}+\sqrt{c})}=\frac{(\sqrt{c}+\sqrt{a})-(\sqrt{a}+\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

i.e., if $\frac{(\sqrt{b}-\sqrt{a})}{(\sqrt{c}+\sqrt{a})(\sqrt{b}+\sqrt{c})}=\frac{(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

i.e., if $\frac{(\sqrt{b}-\sqrt{a})}{(\sqrt{b}+\sqrt{c})}=\frac{(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})}$

i.e., if $b-a=c-b$

i.e., if $2b=a+c$, which is true by (i).

$\therefore$ *a, b, c* are in AP $\Rightarrow$ $\frac{1}{(\sqrt{b}+\sqrt{c})}, \frac{1}{(\sqrt{c}+\sqrt{a})}, \frac{1}{(\sqrt{a}+\sqrt{b})}$ are in AP.

EXAMPLE 3 *If a, b, c are in AP, show that*

$$[(b+c)^2-a^2], [(c+a)^2-b^2], [(a+b)^2-c^2] \text{ are in AP.}$$

SOLUTION Since *a, b, c* are in AP, we have

$$2b=(a+c). \qquad \ldots \text{(i)}$$

Now, $[(b+c)^2-a^2], [(c+a)^2-b^2], [(a+b)^2-c^2]$ will be in AP

if $[(c+a)^2-b^2]-[(b+c)^2-a^2]=[(a+b)^2-c^2]-[(c+a)^2-b^2]$

i.e., if $(a+b+c)(c+a-b)-(a+b+c)(b+c-a)$
$=(a+b+c)(a+b-c)-(a+b+c)(c+a-b)$

i.e., if $(a+b+c)[(c+a-b)-(b+c-a)]$
$=(a+b+c)[(a+b-c)-(c+a-b)]$

i.e., if $2(a-b)=2(b-c)$

i.e., if $(a-b)=(b-c)$

i.e., if $2b = (a + c)$, which is true by (i).

$\therefore$ a, b, c are in AP

$\Rightarrow$ $[(b+c)^2 - a^2], [(c+a)^2 - b^2], [(a+b)^2 - c^2]$ are in AP.

EXAMPLE 4 *If a, b, c are in AP, show that*

(i) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ *are in AP.*

(ii) $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ *are in AP.*

SOLUTION (i) Since a, b, c are in AP, the terms obtained by dividing each term of this AP by abc are also in AP.

Consequently, $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in AP.

(ii) a, b, c are in AP

$\Rightarrow$ $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in AP [dividing each term by abc]

$\Rightarrow$ $\left(\frac{ab+bc+ca}{bc}\right), \left(\frac{(ab+bc+ca}{ca}\right), \left(\frac{ab+bc+ca}{ab}\right)$ are in AP

[multiplying each term by $(ab + bc + ca)$]

$\Rightarrow$ $\left(\frac{ab+bc+ca}{bc}-1\right), \left(\frac{ab+bc+ca}{ca}-1\right), \left(\frac{ab+bc+ca}{ab}-1\right)$ are in AP

[adding –1 to each term]

$\Rightarrow$ $\frac{(ac+ab)}{bc}, \frac{(ab+bc)}{ca}, \frac{(bc+ca)}{ab}$ are in AP

$\Rightarrow$ $a\left(\frac{c+b}{bc}\right), b\left(\frac{a+c}{ca}\right), c\left(\frac{b+a}{ab}\right)$ are in AP

$\Rightarrow$ $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in AP.

EXAMPLE 5 *If a^2, b^2, c^2 are in AP, prove that*

$\frac{1}{(b+c)}, \frac{1}{(c+a)}, \frac{1}{(a+b)}$ *are in AP.*

SOLUTION $\frac{1}{(b+c)}, \frac{1}{(c+a)}, \frac{1}{(a+b)}$ will be in AP

if $\frac{1}{(c+a)} - \frac{1}{(b+c)} = \frac{1}{(a+b)} - \frac{1}{(c+a)}$

i.e., if $\frac{(b+c)-(c+a)}{(c+a)(b+c)} = \frac{(c+a)-(a+b)}{(a+b)(c+a)}$

i.e., if $\frac{(b-a)}{(c+a)(b+c)} = \frac{(c-b)}{(a+b)(c+a)}$

i.e., if $\frac{(b-a)}{(b+c)} = \frac{(c-b)}{(a+b)}$

i.e., if $(b-a)(b+a)=(c+b)(c-b)$

i.e., if $b^2-a^2=c^2-b^2$

i.e., if a^2, b^2, c^2 are in AP.

Hence, a^2, b^2, c^2 are in AP $\Rightarrow \frac{1}{(b+c)}, \frac{1}{(c+a)}, \frac{1}{(a+b)}$ are in AP.

EXAMPLE 6 *If a^2, b^2, c^2 are in AP, prove that*

$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ *are in AP.*

SOLUTION $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ will be in AP

if $\frac{a}{b+c}+1, \frac{b}{c+a}+1, \frac{c}{a+b}+1$ are in AP

[on adding 1 to each term]

i.e., if $\frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b}$ are in AP

i.e., if $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in AP

[on dividing each term by $(a+b+c)$]

i.e., if $\frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a}$

i.e., if $\frac{(b+c)-(c+a)}{(c+a)(b+c)}=\frac{(c+a)-(a+b)}{(a+b)(c+a)}$

i.e., if $\frac{(b-a)}{(b+c)}=\frac{(c-b)}{(a+b)}$

i.e., if $(b-a)(b+a)=(c-b)(c+b)$

i.e., if $b^2-a^2=c^2-b^2$

i.e., if a^2, b^2, c^2 are in AP.

Hence, a^2, b^2, c^2 are in AP $\Rightarrow \frac{a}{(b+c)}, \frac{b}{(c+a)}, \frac{c}{(a+b)}$ are in AP.

EXAMPLE 7 *Show that $(a^2+ab+b^2), (c^2+ac+a^2)$ and (b^2+bc+c^2) are in AP, if a, b, c are in AP.*

SOLUTION The terms $(a^2+ab+b^2), (c^2+ac+a^2), (b^2+bc+c^2)$ will be in AP

if $(c^2+ac+a^2)-(a^2+ab+b^2)=(b^2+bc+c^2)-(c^2+ac+a^2)$

i.e., if $c^2+ac-ab-b^2=b^2+bc-ac-a^2$

i.e., if $(a^2+c^2+2ac)-b^2=b^2+bc+ab$

i.e., if $(a+c)^2-b^2=b(a+b+c)$

i.e., if $(a+c+b)(a+c-b)=b(a+b+c)$

i.e., if $a+c-b=b$

i.e., if $a+c=2b$

i.e., if *a, b, c* are in AP.

Hence, *a, b, c* are in AP $\Rightarrow (a^2+ab+b^2), (c^2+ac+a^2)$ and (b^2+bc+c^2)

are in AP.

EXAMPLE 8 *If* $(b-c)^2, (c-a)^2, (a-b)^2$ *are in AP, prove that* $\frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)}$ *are in AP.*

SOLUTION $(b-c)^2, (c-a)^2, (a-b)^2$ are in AP

$\Rightarrow \quad (c-a)^2-(b-c)^2=(a-b)^2-(c-a)^2$

$\Rightarrow \quad (b-a)(2c-a-b)=(c-b)(2a-b-c).$ ... (i)

Now, $\frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)}$ will be in AP

if $\frac{1}{(c-a)}-\frac{1}{(b-c)}=\frac{1}{(a-b)}-\frac{1}{(c-a)}$

i.e., if $\frac{(b-c)-(c-a)}{(c-a)(b-c)}=\frac{(c-a)-(a-b)}{(a-b)(c-a)}$

i.e., if $\frac{(a+b-2c)}{(c-a)(b-c)}=\frac{(b+c-2a)}{(a-b)(c-a)}$

i.e., if $(a-b)(a+b-2c)=(b-c)(b+c-2a)$

i.e., if $(b-a)(2c-a-b)=(c-b)(2a-b-c)$, which is true by (i).

Hence, $(b-c)^2, (c-a)^2, (a-b)^2$ are in AP

$\Rightarrow \quad \frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)}$ are in AP.

EXAMPLE 9 *If* $(a^2+2bc), (b^2+2ac), (c^2+2ab)$ *are in AP, show that* $\frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)}$ *are in AP.*

SOLUTION $a^2+2bc, b^2+2ac, c^2+2ab$ are in AP

$\Rightarrow \quad (a^2+2bc)-(ab+bc+ac), (b^2+2ac)-(ab+bc+ac), (c^2+2ab)$
$-(ab+bc+ac)$ are in AP

[by adding $(ab+bc+ac)$ to each term]

$\Rightarrow \quad (a^2+bc-ab-ac), (b^2+ac-ab-bc), (c^2+ab-bc-ac)$ are in AP

$\Rightarrow \quad (a-b)(a-c), (b-c)(b-a), (c-b)(c-a)$ are in AP

$\Rightarrow \quad -(a-b)(c-a), -(a-b)(b-c), -(b-c)(c-a)$ are in AP

$\Rightarrow \quad \frac{-1}{(b-c)}, \frac{-1}{(c-a)}, \frac{-1}{(a-b)}$ are in AP

[on dividing each term by $(a-b)(b-c)(c-a)$]

$\Rightarrow \quad \frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)}$ are in AP

[on multiplying each term by –1].

Hence, the result follows.

EXAMPLE 10 *If* $\frac{(b+c-a)}{a}, \frac{(c+a-b)}{b}, \frac{(a+b-c)}{c}$ *are in AP, prove that* $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ *are in AP.*

SOLUTION $\frac{(b+c-a)}{a}, \frac{(c+a-b)}{b}, \frac{(a+b-c)}{c}$ are in AP

$\Rightarrow \quad \left\{\frac{(b+c-a)}{a}+2\right\}, \left\{\frac{(c+a-b)}{b}+2\right\}, \left\{\frac{(a+b-c)}{c}+2\right\}$ are in AP

[adding 2 to each term]

$\Rightarrow \quad \frac{(a+b+c)}{a}, \frac{(a+b+c)}{b}, \frac{(a+b+c)}{c}$ are in AP

$\Rightarrow \quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP [on dividing each term by $(a+b+c)$].

EXAMPLE 11 *If $a^2(b+c), b^2(c+a), c^2(a+b)$ are in AP then show that either a, b, c are in AP or $ab+bc+ca=0$.*

SOLUTION $a^2(b+c), b^2(c+a), c^2(a+b)$ are in AP

$\Rightarrow \quad b^2(c+a)-a^2(b+c)=c^2(a+b)-b^2(c+a)$

$\Rightarrow \quad (b^2a-a^2b)+(b^2c-a^2c)=(c^2b-b^2c)+(c^2a-b^2a)$

$\Rightarrow \quad ab(b-a)+c(b^2-a^2)=bc(c-b)+a(c^2-b^2)$

$\Rightarrow \quad (b-a)(ab+bc+ca)=(c-b)(ab+bc+ca)$

$\Rightarrow \quad (ab+bc+ca)\{(b-a)-(c-b)\}=0$

$\Rightarrow \quad (ab+bc+ca)(2b-a-c)=0$

$\Rightarrow \quad ab+bc+ca=0 \text{ or } 2b-a-c=0$

$\Rightarrow \quad ab+bc+ca=0 \text{ or } 2b=a+c$

$\Rightarrow \quad ab+bc+ca=0 \text{ or } a, b, c$ are in AP.

EXAMPLE 12 *If a, b, c are in AP, prove that $(a-c)^2=4(b^2-ac)$.*

SOLUTION Since *a, b, c* are in AP, we have $b=\frac{1}{2}(a+c)$.

$\therefore \quad \text{RHS}=4(b^2-ac)=4\times\left\{\frac{1}{4}(a+c)^2-ac\right\}$

$=(a+c)^2-4ac=(a-c)^2=\text{LHS}.$

Hence, $(a-c)^2=4(b^2-ac)$.

EXAMPLE 13 *If a, b, c are in AP, prove that $a^3+4b^3+c^3=3b(a^2+c^2)$.*

SOLUTION Since *a, b, c* are in AP, we have $b=\frac{1}{2}(a+c)$.

$\therefore \quad \text{RHS}=3\cdot\frac{1}{2}(a+c)(a^2+c^2)=\frac{3}{2}(a^3+c^3+ac^2+a^2c)$

and $\text{LHS}=a^3+4\cdot\left(\frac{a+c}{2}\right)^3+c^3=\frac{3}{2}(a^3+c^3+ac^2+a^2c).$

$\therefore \quad \text{LHS}=\text{RHS}.$

EXERCISE 11E

1. If *a, b, c* are in AP, prove that

(i) $(a-c)^2=4(a-b)(b-c)$ (ii) $a^2+c^2+4ac=2(ab+bc+ca)$

(iii) $a^3+c^3+6abc=8b^3$

2. If a, b, c are in AP, show that

$$(a+2b-c)(2b+c-a)(c+a-b)=4abc.$$

3. If a, b, c are in AP, show that

(i) $(b+c-a), (c+a-b), (a+b-c)$ are in AP.

(ii) $(bc-a^2), (ca-b^2), (ab-c^2)$ are in AP.

4. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP, prove that

(i) $\frac{(b+c)}{a}, \frac{(c+a)}{b}, \frac{(a+b)}{c}$ are in AP.

(ii) $\frac{(b+c-a)}{a}, \frac{(c+a-b)}{b}, \frac{(a+b-c)}{c}$ are in AP.

5. If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in AP, prove that $a^2(b+c), b^2(c+a), c^2(a+b)$ are in AP.

6. If a, b, c are in AP, show that $\frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab}$ are also in AP.

HINTS TO SOME SELECTED QUESTIONS

4. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

$\Rightarrow$ $\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in AP

$\Rightarrow$ $1+\frac{b+c}{a}, 1+\frac{c+a}{b}, 1+\frac{a+b}{c}$ are in AP

$\Rightarrow$ $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in AP. This proves (i).

$\therefore$ $\frac{b+c}{a}-1, \frac{c+a}{b}-1, \frac{a+b}{c}-1$ are in AP

$\Rightarrow$ $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in AP. This proves (ii).

5. $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in AP

$\Rightarrow$ $\frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab}$ are in AP

$\Rightarrow$ $\frac{a^2(b+c)}{abc}, \frac{b^2(c+a)}{abc}, \frac{c^2(a+b)}{abc}$ are in AP

$\Rightarrow$ $a^2(b+c), b^2(c+a), c^2(a+b)$ are in AP.

6. $\frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab}$ will be in AP

if $\frac{a(b+c)}{bc}+1, \frac{b(c+a)}{ca}+1, \frac{c(a+b)}{ab}+1$ are in AP

i.e., if $\frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ca}, \frac{ab+bc+ca}{ab}$ are in AP

i.e., if $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in AP

i.e., if $\frac{abc}{bc}, \frac{abc}{ca}, \frac{abc}{ab}$ are in AP

i.e., if a, b, c are in AP.

EXERCISE 11F

Very-Short-Answer Questions

1. If the sum of n terms of an AP is given by $S_n = (2n^2 + 3n)$ then find its common difference.
2. If 9 times the 9th term of an AP is equal to 13 times the 13th term, show that its 22nd term is 0.
3. In an AP it is given that $S_n = qn^2$ and $S_m = qm^2$. Prove that $S_q = q^3$.
4. Find three arithmetic means between 6 and –6.
5. The 9th term of an AP is 0. Prove that its 29th term is double the 19th term.
6. How many terms are there in the AP 13, 16, 19, …, 43?
7. Find the 8th term from the end of the AP 7, 9, 11, …, 201.
8. How many 2-digit numbers are divisible by 7?
9. If 7th and 13th terms of an AP be 34 and 64 respectively then find its 18th term.
10. What is the 10th common term between the APs 3, 7, 11, 15, 19, … and 1, 6, 11, 16, …?
11. The first and last terms of an AP are 1 and 11 respectively. If the sum of its terms is 36, find the number of terms.
12. In an AP, the pth term is q and $(p+q)$th term is 0. Show that its qth term is p.
13. If $\frac{3+5+7+9+\ldots \text{ up to 35 terms}}{5+8+11+\ldots \text{ up to } n \text{ terms}} = 7$, find the value of n.
14. Write the sum of first n even natural numbers.
15. Write the sum of first n odd natural numbers.
16. The sum of n terms of an AP is $\frac{1}{2}an^2 + bn$. Find the common difference.
17. If the sums of n terms of two APs are in the ratio $(2n+3):(3n+2)$, find the ratio of their 10th terms.

ANSWERS (EXERCISE 11F)

1. 4	**4.** 3, 0, –3	**6.** 11	**7.** 197	**8.** 13
9. 89	**10.** 191	**11.** 6	**13.** $n = 10$	**14.** $n(n+1)$
15. n^2	**16.** a	**17.** 41 : 59		

1. $T_n = (S_n - S_{n-1}) = (2n^2 + 3n) - \{2(n-1)^2 + 3(n-1)\}$

$= (2n^2 + 3n) - (2n^2 - n - 1) = 4n + 1.$

$\therefore \quad d = (T_2 - T_1) = (4 \times 2 + 1) - (4 \times 1 + 1) = (9 - 5) = 4.$

2. $9(a + 8d) = 13(a + 12d) \Rightarrow 4a + 84d = 0 \Rightarrow a + 21d = 0 \Rightarrow T_{22} = 0.$

3. $S_n = qn^2 \Rightarrow S_1 = q$ and $S_2 = 4q$

$\Rightarrow T_1 = q$ and $T_1 + T_2 = 4q \Rightarrow T_1 = q$ and $T_2 = 3q.$

$\therefore \quad S_q = \frac{q}{2} \times [2q + (q-1) \times 2q] = \left(\frac{q}{2} \times 2q^2\right) = q^3.$

4. Let the required AM be x_1, x_2, x_3. Then,

$6, x_1, x_2, x_3, -6$ are in AP.

Now, $T_5 = -6 \Rightarrow a + 4d = -6 \Rightarrow 6 + 4d = -6 \Rightarrow 4d = -12 \Rightarrow d = -3.$

$\therefore \quad x_1 = 6 + d = (6 - 3) = 3, x_2 = 6 + 2d = (6 - 6) = 0, x_3 = 6 + 3d = (6 - 9) = -3.$

5. $T_9 = 0 \Rightarrow a + 8d = 0 \Rightarrow a = -8d.$

$T_{19} = a + 18d = -8d + 18d = 10d$ and $T_{29} = a + 28d = -8d + 28d = 20d.$

$\therefore \quad T_{29} = 2 \times T_{19}.$

6. Let it contain n terms. Then, $T_n = 43 \Rightarrow a + (n-1)d = 43.$

$\therefore \quad 13 + (n-1) \times 3 = 43 \Rightarrow n = 11.$

7. Here, $a = 7, d = (9 - 7) = 2$ and $l = 201.$

Let $T_n = 201$. Then, $a + (n-1)d = 201 \Rightarrow 7 + (n-1) \times 2 = 201 \Rightarrow n = 98.$

$\therefore \quad$ 8th term from the end $= l - (n-1)d = (201 - 7 \times 2) = (201 - 14) = 197.$

8. 2-digit numbers divisible by 7 are 14, 21, 28, …, 98.

Let their number be n. Then, $T_n = 98.$

$\therefore \quad 14 + (n-1) \times 7 = 98 \Rightarrow n = 13.$

9. $a + 6d = 34 \quad \ldots$ (i) and $a + 12d = 64. \quad \ldots$ (ii)

On solving (i) and (ii), we get $a = 4$ and $d = 5.$

$\therefore \quad T_{18} = a + 17d = (4 + 17 \times 5) = 89.$

10. The first common term = 11.

Common difference of new AP = LCM (4, 5) = 20.

So, the new AP is 11, 31, 51, … .

Its 10th term = $(11 + 9 \times 20) = 191.$

11. $\frac{n}{2}(a + l) = 36 \Rightarrow \frac{n}{2} \times (1 + 11) = 36 \Rightarrow n = 6.$

13. $\frac{\frac{35}{2} \cdot \{6 + 34 \times 2\}}{\frac{n}{2} \cdot \{10 + (n-1) \times 3\}} = 7 \Rightarrow 3n^2 + 7n - 370 = 0$

$\Rightarrow \quad (3n + 37)(n - 10) = 0 \Rightarrow n = 10.$

14. Given sum $= 2 + 4 + 6 + 8 + \ldots + 2n$

$= \frac{n}{2}(2 + 2n) = n(n + 1).$

15. Given sum $= 1 + 3 + 5 + \ldots$ to n terms

$= \frac{n}{2} \times [2 \times 1 + (n-1) \times 2] = n^2.$

16. $T_n = (S_n - S_{n-1})$

$$= \left\{\frac{1}{2}an^2 + bn\right\} - \left\{\frac{1}{2}a(n-1)^2 + b(n-1)\right\}$$

$$= \left\{\frac{1}{2}an^2 + bn\right\} - \left\{\frac{1}{2}an^2 - an + \frac{1}{2}a + bn - b\right\} = \left(an + b - \frac{1}{2}a\right).$$

$$T_{n-1} = a(n-1) + b - \frac{1}{2}a = \left(an + b - \frac{1}{2}a\right) - a.$$

$$\therefore \quad d = (T_n - T_{n-1}) = a.$$

17. $$\frac{\frac{n}{2}\cdot[2a_1 + (n-1)d_1]}{\frac{n}{2}\cdot[2a_2 + (n-1)d_2]} = \frac{2n+3}{3n+2} \Rightarrow \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{2n+3}{3n+2}.$$

$$\therefore \quad \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{2n+3}{3n+2} \Rightarrow \frac{a_1 + 9d_1}{a_2 + 9d_2} = \frac{2\times 19 + 3}{3\times 19 + 2} \qquad \left[\text{putting } \frac{n-1}{2} = 9 \Rightarrow n = 19\right]$$

$$\Rightarrow \frac{a_1 + 9d_1}{a_2 + 9d_2} = \frac{41}{59}.$$

Hence, the ratio of their 10th terms is 41 : 59.

REVIEW OF FACTS AND FORMULAE

1. (i) **Sequence** A succession of numbers arranged in a definite order according to a certain given rule is called a sequence.

The number occurring at the *n*th place of a sequence is called its *n*th term, denoted by a_n.

(ii) By adding the terms of a sequence, we get a series.

(iii) A series is said to be finite or infinite according as the number of terms in it is finite or infinite respectively.

2. (i) **Progressions** Sequences following certain patterns are called progressions.

(ii) **Arithmetic Progression (AP)**

The progression of the form $a, (a+d), (a+2d), \ldots$ is called an AP with first term $= a$ and common difference $= d$.

We have, $T_n = a + (n-1)d$, called the general term.

Clearly, $T_n - T_{n-1} = d$.

(iii) If *l* is the last term of the AP then *n*th term from the end $= l - (n-1)d$.

(iv) If an AP contains *n* terms then

*p*th term from the end = $(n - p + 1)$th term from the beginning.

3. Sum to *n* terms of an AP is given by

(i) $S_n = \frac{n}{2}\cdot\{2a + (n-1)d\}$

(ii) $S_n = \frac{n}{2}(a + l)$

(iii) $S_1 = T_1, (S_2 - S_1) = T_2, S_n - S_{n-1} = T_n$

4. (i) 3 numbers in AP are taken as $(a-d), a, (a+d)$.

(ii) 4 numbers in AP are taken as $(a-3d), (a-d), (a+d), (a+3d)$.

(iii) 5 numbers in AP are taken as $(a-2d), (a-d), a, (a+d), (a+2d)$.

5. If $a_1, a_2, a_3, a_4, \ldots$ are in AP and k is a nonzero number then

(i) $(a_1+k), (a_2+k), (a_3+k), (a_4+k), \ldots$ are in AP.

(ii) $(a_1-k), (a_2-k), (a_3-k), (a_4-k), \ldots$ are in AP.

(iii) $a_1k, a_2k, a_3k, a_4k, \ldots$ are in AP.

(iv) $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \frac{a_4}{k}, \ldots$ are in AP.

6. (i) **Arithmetic Mean (AM)** If a, A, b are in AP then A is the arithmetic mean between a and b and $A = \frac{1}{2}(a+b)$.

(ii) If $a, A_1, A_2, \ldots, A_n, b$ are in AP then $A_1, A_2, \ldots, A_n$ are n arithmetic means between a and b.

SUM OF FIRST *n* NATURAL NUMBERS

THEOREM 1 *Prove that* $(1+2+3+\ldots+n)=\frac{1}{2}n(n+1)$

i.e., $\sum_{k=1}^{n} k=\frac{1}{2}n(n+1)$.

PROOF Consider the series $(1+2+3+\ldots+n)$.

This is an arithmetic series in which $a=1, d=1$ and $l=n$.

$\therefore \quad S_n=\frac{n}{2}(1+n) \Rightarrow S_n=\frac{1}{2}n(n+1) \qquad \left[\because S_n=\frac{1}{2}(a+l)\right]$.

Hence, $\sum_{k=1}^{n} k=\frac{1}{2}n(n+1)$.

SUM OF THE SQUARES OF FIRST *n* NATURAL NUMBERS

THEOREM 2 *Prove that* $(1^2+2^2+3^2+\ldots+n^2)=\frac{1}{6}n(n+1)(2n+1)$

i.e., $\sum_{k=1}^{n} k^2=\frac{1}{6}n(n+1)(2n+1)$.

PROOF Let $S_n=1^2+2^2+3^2+\ldots+n^2$.

Consider the identity: $k^3-(k-1)^3=3k^2-3k+1$. ... (i)

Putting $k=1, 2, 3, \ldots, (n-1), n$ successively in (i), we get

$1^3-0^3=3\cdot 1^2-3\cdot 1+1;$

$2^3-1^3=3\cdot 2^2-3\cdot 2+1;$

$3^3-2^3=3\cdot 3^2-3\cdot 3+1;$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$(n-1)^3-(n-2)^3=3\cdot(n-1)^2-3\cdot(n-1)+1;$

$n^3-(n-1)^3=3\cdot n^2-3\cdot n+1.$

Adding columnwise, we get

$(n^3-0^3)=3(1^2+2^2+3^2+\ldots+n^2)-3(1+2+3+\ldots+n)+n$

$\Rightarrow \quad n^3=3\cdot\sum_{k=1}^{n} k^2-3\cdot\frac{1}{2}n(n+1)+n \quad \left[\because (1+2+3+\ldots+n)=\frac{1}{2}n(n+1)\right]$

$\Rightarrow \quad 3\left(\sum_{k=1}^{n} k^2\right)=n^3+\frac{3}{2}n(n+1)-n=\frac{1}{2}(2n^3+3n^2+n)$

$$\Rightarrow \quad \left[\sum_{k=1}^{n} k^2\right] = \frac{1}{6}(2n^3+3n^2+n) = \frac{1}{6}n(2n^2+3n+1) = \frac{1}{6}n(n+1)(2n+1).$$

Hence, $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

SUM OF THE CUBES OF FIRST *n* NATURAL NUMBERS

THEOREM 3 *Prove that* $(1^3+2^3+3^3+\ldots+n^3) = \left\{\frac{1}{2}n(n+1)\right\}^2$

i.e., $\sum_{k=1}^{n} k^3 = \left\{\frac{1}{2}n(n+1)\right\}^2$.

PROOF Let $S_n = 1^3+2^3+3^3+\ldots+n^3$.

Consider the identity: $k^4-(k-1)^4 = 4k^3-6k^2+4k-1$. … (i)

Putting $k = 1, 2, 3, \ldots, (n-1), n$ successively in (i), we get

$$1^4-0^4 = 4\cdot 1^3-6\cdot 1^2+4\cdot 1-1$$
$$2^4-1^4 = 4\cdot 2^3-6\cdot 2^2+4\cdot 2-1$$
$$3^4-2^4 = 4\cdot 3^3-6\cdot 3^2+4\cdot 3-1$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$(n-1)^4-(n-2)^4 = 4\cdot(n-1)^3-6\cdot(n-1)^2+4\cdot(n-1)-1$$
$$n^4-(n-1)^4 = 4\cdot n^3-6\cdot n^2+4\cdot n-1.$$

Adding columnwise, we get

$$(n^4-0^4) = 4\cdot\{1^3+2^3+3^3+\ldots+n^3\}-6\cdot\{1^2+2^2+3^2+\ldots+n^2)$$
$$+4\cdot\{1+2+3+\ldots+n\}-(1+1+1+\ldots \text{to } n \text{ terms})$$

$$\Rightarrow \quad n^4 = 4\cdot\left(\sum_{k=1}^{n} k^3\right)-6\cdot\left(\sum_{k=1}^{n} k^2\right)+4\cdot\left(\sum_{k=1}^{n} k\right)-n$$

$$\Rightarrow \quad n^4 = 4\cdot\left(\sum_{k=1}^{n} k^3\right)-6\cdot\frac{1}{6}n(n+1)(2n+1)+4\cdot\frac{1}{2}n(n+1)-n$$

$$\left\{\because \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \text{ and } \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)\right\}$$

$$\Rightarrow \quad 4\cdot\left(\sum_{k=1}^{n} k^3\right) = n^4+n(n+1)(2n+1)-2n(n+1)+n$$

$$\Rightarrow \quad \left(\sum_{k=1}^{n} k^3\right) = \frac{1}{4}\cdot(n^4+2n^3+n^2) = \frac{1}{4}n^2(n^2+2n+1) = \left\{\frac{1}{2}n(n+1)\right\}^2$$

$$\therefore \quad \left(\sum_{k=1}^{n} k^3\right) = \left\{\frac{1}{2}n(n+1)\right\}^2$$

i.e., $(1^3+2^3+3^3+\ldots+n^3) = \left\{\frac{1}{2}n(n+1)\right\}^2$.

SUM OF THE CUBES OF FIRST *n* ODD NATURAL NUMBERS

THEOREM 4 *Prove that* $[1^3+3^3+5^3+\ldots(2n-1)^3] = n^2(2n^2-1)$

i.e., $\sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1)$.

PROOF Let $S_n = 1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$.

Then, $S_n = \sum_{k=1}^{n} (2k-1)^3$

$= \sum_{k=1}^{n} (8k^3 - 12k^2 + 6k - 1)$

$= 8\left(\sum_{k=1}^{n} k^3\right) - 12\left(\sum_{k=1}^{n} k^2\right) + 6\left(\sum_{k=1}^{n} k\right) - n$ [$\because$ $1 + 1 + \ldots$ to n terms $= n$]

$= 8 \cdot \frac{1}{4} n^2 (n+1)^2 - 12 \cdot \frac{1}{6} n(n+1)(2n+1) + 6 \cdot \frac{1}{2} n(n+1) - n$

$= 2n^2 (n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$

$= n(n+1) \cdot \{2n(n+1) - 2(2n+1) + 3\} - n$

$= n(n+1)(2n^2 - 2n + 1) - n = n \cdot \{(n+1)(2n^2 - 2n + 1) - 1\}$

$= n(2n^3 - n) = n^2 (2n^2 - 1)$.

$\therefore \quad \{1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3\} = n^2 (2n^2 - 1)$.

SUMMARY

(i) $(1 + 2 + 3 + \ldots + n) = \frac{1}{2} n(n+1)$

i.e., $\left(\sum_{k=1}^{n} k\right) = \frac{1}{2} n(n+1)$.

(ii) $(1^2 + 2^2 + 3^2 + \ldots + n^2) = \frac{1}{6} n(n+1)(2n+1)$

i.e., $\left(\sum_{k=1}^{n} k^2\right) = \frac{1}{6} n(n+1)(2n+1)$.

(iii) $(1^3 + 2^3 + 3^3 + \ldots + n^3) = \left\{\frac{1}{2} n(n+1)\right\}^2$

i.e., $\left(\sum_{k=1}^{n} k^3\right) = \left\{\frac{1}{2} n(n+1)\right\}^2$.

(iv) $\{1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3\} = n^2 (2n^2 - 1)$

i.e., $\sum_{k=1}^{n} (2k-1)^3 = n^2 (2n^2 - 1)$.

SOLVED EXAMLES

EXAMPLE 1 *If S_1, S_2 and S_3 are the sums of first n natural numbers, their squares and their cubes respectively then show that $9S_2^2 = S_3(1 + 8S_1)$.*

SOLUTION We have

$S_1 = (1 + 2 + 3 + \ldots + n) \Rightarrow S_1 = \frac{1}{2} n(n+1)$;

$S_2 = (1^2 + 2^2 + 3^2 + \ldots + n^2) \Rightarrow S_2 = \frac{1}{6} n(n+1)(2n+1)$;

and $S_3 = (1^2 + 2^3 + 3^3 + \ldots + n^3) \Rightarrow S_3 = \frac{1}{4} n^2 (n+1)^2$.

$$\therefore \quad 9S_2^2 = 9 \times \frac{1}{36} \cdot n^2(n+1)^2(2n+1)^2 = \frac{1}{4}n^2(n+1)^2(2n+1)^2.$$

And, $S_3(1+8S_1) = \frac{1}{4}n^2(n+1)^2 \cdot \left\{1 + 8 \cdot \frac{1}{2}n(n+1)\right\}$

$$= \frac{1}{4}n^2(n+1)^2(4n^2+4n+1) = \frac{1}{4}n^2(n+1)^2(2n+1)^2.$$

Hence, $9S_2^2 = S_3(1+8S_1)$.

EXAMPLE 2 *Find the sum to n terms of the series whose nth term is* $n(n+3)$.

SOLUTION We have, $T_k = k(k+3) = (k^2+3k)$.

$\therefore$ sum to *n* terms is given by

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n}(k^2+3k) = \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k$$

$$= \frac{1}{6}n(n+1)(2n+1) + 3 \cdot \frac{1}{2}n(n+1) = \frac{1}{6}\{n(n+1)(2n+1) + 9n(n+1)\}$$

$$= \frac{1}{6}n(n+1)(2n+1+9) = \frac{1}{6}n(n+1) \cdot 2(n+5) = \frac{1}{3}n(n+1)(n+5).$$

Hence, the required sum is $\frac{1}{3}n(n+1)(n+5)$.

EXAMPLE 3 *Find the sum to n terms of the series whose nth term is* $(2n-1)^2$.

SOLUTION We have, $T_k = (2k-1)^2 = (4k^2-4k+1)$.

$\therefore$ sum to *n* terms is given by

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n}(4k^2-4k+1) = 4 \cdot \sum_{k=1}^{n} k^2 - 4 \cdot \sum_{k=1}^{n} k + n$$

$$[\because \ 1+1+\ldots n \text{ times} = n]$$

$$= 4 \cdot \frac{1}{6}n(n+1)(2n+1) - 4 \cdot \frac{1}{2}n(n+1) + n$$

$$= \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n$$

$$= \frac{1}{3} \cdot \{2n(n+1)(2n+1) - 6n(n+1) + 3n\}$$

$$= \frac{1}{3}[n(n+1)\{2(2n+1) - 6\} + 3n] = \frac{1}{3} \cdot [4n(n+1)(n-1) + 3n]$$

$$= \frac{1}{3}(4n^3 - n) = \frac{1}{3}n(4n^2-1).$$

Hence, the required sum is $\frac{1}{3}n(4n^2-1)$.

EXAMPLE 4 *Find the sum to n terms of the series whose nth term is* (n^2+2^n).

SOLUTION We have, $T_k = (k^2+2^k)$.

$\therefore$ $S_n = \sum_{k=1}^{n} T_k$

$$= \sum_{k=1}^{n}(k^2+2^k) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k$$

$$= \frac{1}{6}n(n+1)(2n+1) + (2 + 2^2 + 2^3 + \ldots + 2^n)$$

$$\left[\because \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)\right]$$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{2(2^n - 1)}{(2-1)} \quad [\because\ 2 + 2^2 + \ldots + 2^n \text{ is a GP}]$$

$$= \frac{1}{6}n(n+1)(2n+1) + 2(2^n - 1).$$

Hence, the required sum is $\frac{1}{6}n(n+1)(2n+1) + 2(2^n - 1)$.

EXAMPLE 5 *Sum the series* $3 \cdot 8 + 6 \cdot 11 + 9 \cdot 14 + \ldots$ *to n terms.*

SOLUTION We have

$$T_k = (k\text{th term of } 3, 6, 9, \ldots) \times (k\text{th term of } 8, 11, 14, \ldots)$$

$$= \{3 + (k-1) \times 3\} \times \{8 + (k-1) \times 3\} = 3k(3k+5)$$

$$= (9k^2 + 15k).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (9k^2 + 15k) = 9\left(\sum_{k=1}^{n} k^2\right) + 15\left(\sum_{k=1}^{n} k\right)$$

$$= 9 \cdot \left\{\frac{1}{6}n(n+1)(2n+1)\right\} + 15 \cdot \left\{\frac{1}{2}n(n+1)\right\}$$

$$= \frac{3}{2}n(n+1)\{(2n+1) + 5\} = 3n(n+1)(n+3).$$

Hence, the required sum is $3n(n+1)(n+3)$.

EXAMPLE 6 *Sum the series* $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots$ *to n terms.*

SOLUTION We have

$$T_k = (k\text{th term of } 1, 2, 3, \ldots) \times (k\text{th term of } 2, 3, 4, \ldots) \times (k\text{th term of } 3, 4, 5, \ldots)$$

$$= \{1 + (k-1) \times 1\} + \{2 + (k-1) \times 1\} \times \{3 + (k-1) \times 1\}$$

$$= k(k+1)(k+2) = (k^3 + 3k^2 + 2k).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (k^3 + 3k^2 + 2k) = \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k$$

$$= \frac{1}{4}n^2(n+1)^2 + 3 \cdot \frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1)$$

$$\left\{\because \sum_{k=1}^{n} k^3 = \left\{\frac{1}{2}n(n+1)\right\}^2, \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1), \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)\right\}$$

$$= \frac{1}{4}n(n+1)\{n(n+1) + 2(2n+1) + 4\}$$

$$= \frac{1}{4}n(n+1)(n^2+5n+6) = \frac{1}{4}n(n+1)(n+2)(n+3).$$

Hence, the required sum is $\frac{1}{4}n(n+1)(n+2)(n+3)$.

EXAMPLE 7 *Find the sum of the series* $1^2+3^2+5^2+\ldots$ *to n terms.*

SOLUTION We have

$$T_k = k\text{th term of } (1^2+3^2+5^2+\ldots)$$
$$= \{1+(k-1)\times 2\}^2 = (2k-1)^2 = (4k^2-4k+1).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n}(4k^2-4k+1)$$
$$= 4\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + (1+1+\ldots n \text{ times})$$
$$= 4\cdot\frac{1}{6}n(n+1)(2n+1) - 4\cdot\frac{1}{2}n(n+1) + n$$
$$= \frac{n}{3}\cdot\{2(n+1)(2n+1) - 6(n+1) + 3\} = \frac{n}{3}(4n^2-1).$$

Hence, the required sum is $\frac{n}{3}(4n^2-1)$.

EXAMPLE 8 *Find the sum* $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots$ *to n terms.*

SOLUTION We have

$$T_k = (1^2+2^2+3^2+\ldots+k^2) = \left\{\frac{1}{6}k(k+1)(2k+1)\right\} = \frac{1}{6}(2k^3+3k^2+k).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$
$$= \frac{1}{6}\cdot\sum_{k=1}^{n}(2k^3+3k^2+k) = \frac{1}{6}\cdot 2\cdot\sum_{k=1}^{n}k^3 + \frac{1}{6}\cdot 3\cdot\sum_{k=1}^{n}k^2 + \frac{1}{6}\cdot\sum_{k=1}^{n}k$$
$$= \frac{1}{3}\cdot\sum_{k=1}^{n}k^3 + \frac{1}{2}\cdot\sum_{k=1}^{n}k^2 + \frac{1}{6}\cdot\sum_{k=1}^{n}k$$
$$= \frac{1}{3}\cdot\left\{\frac{1}{2}n(n+1)\right\}^2 + \frac{1}{2}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{1}{6}\cdot\frac{1}{2}n(n+1)$$
$$= \frac{1}{12}n^2(n+1)^2 + \frac{1}{12}n(n+1)(2n+1) + \frac{1}{12}n(n+1)$$
$$= \frac{1}{12}n(n+1)\cdot\{n(n+1)+(2n+1)+1\} = \frac{1}{12}n(n+1)^2(n+2).$$

Hence, the required sum is $\frac{1}{12}n(n+1)^2(n+2)$.

EXAMPLE 9 *Prove that* $\dfrac{1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)} = \dfrac{3n+5}{3n+1}$.

SOLUTION We have

*k*th term of the numerator $= k(k+1)^2 = (k^3+2k^2+k)$.

*k*th term of the denominator $= k^2(k+1) = (k^3+k^2)$.

$$\therefore \quad \text{LHS} = \frac{\sum_{k=1}^{n}(k^3+2k^2+k)}{\sum_{k=1}^{n}(k^3+k^2)} = \frac{\left(\sum_{k=1}^{n}k^3\right)+2\cdot\left(\sum_{k=1}^{n}k^2\right)+\left(\sum_{k=1}^{n}k\right)}{\left(\sum_{k=1}^{n}k^3\right)+\left(\sum_{k=1}^{n}k^2\right)}$$

$$= \frac{\frac{1}{4}n^2(n+1)^2+2\cdot\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)}{\frac{1}{4}n^2(n+1)^2+\frac{1}{6}n(n+1)(2n+1)}$$

$$\left\{\because \left(\sum_{k=1}^{n}k\right)=\frac{1}{2}n(n+1);\left(\sum_{k=1}^{n}k^2\right)=\frac{1}{6}n(n+1)(2n+1)\right.$$

$$\left.\text{and}\left(\sum_{k=1}^{n}k^3\right)=\frac{1}{4}n^2(n+1)^2\right\}$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12}\times\frac{12}{n(n+1)(n+2)(3n+1)}$$

[on simplifying]

$$= \frac{(3n+5)}{(3n+1)} = \text{RHS.}$$

EXAMPLE 10 *Find the sum of n terms of the series*

$$\frac{1}{(2\times5)}+\frac{1}{(5\times8)}+\frac{1}{(8\times11)}+\ldots.$$

SOLUTION We have

$$T_k = \frac{1}{(k\text{th term of } 2, 5, 8, \ldots)\times(k\text{th term of } 5, 8, 11, \ldots)}$$

$$= \frac{1}{\{2+(k-1)\times3\}\times\{5+(k-1)\times3\}}$$

$$= \frac{1}{(3k-1)(3k+2)} = \frac{1}{3}\left\{\frac{1}{(3k-1)}-\frac{1}{(3k+2)}\right\}.$$

$$\therefore \quad T_k = \frac{1}{3}\left\{\frac{1}{(3k-1)}-\frac{1}{(3k+2)}\right\}. \qquad \ldots \text{(i)}$$

Putting $k = 1, 2, 3, \ldots, n$ successively in (i), we get

$$T_1 = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)$$

$$T_2 = \frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)$$

$$T_3 = \frac{1}{3}\left(\frac{1}{8}-\frac{1}{11}\right)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$T_n = \frac{1}{3}\left\{\frac{1}{(3n-1)}-\frac{1}{(3n+2)}\right\}.$$

Adding columnwise, we get

$$S_n = (T_1+T_2+T_3+\ldots+T_n)$$

$$= \frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right) = \frac{n}{2(3n+2)}.$$

EXAMPLE 11 *Find the sum of n terms of the series* $5+11+19+29+41+\ldots$.

SOLUTION Let $S_n = 5+11+19+29+\ldots+T_{n-1}+T_n$

$S_n = \quad 5+11+19+\ldots+T_{n-2}+T_{n-1}+T_n$

On subtraction, we get

$$0 = 5+[6+8+10+12+\ldots \text{ to } (n-1) \text{ terms}]-T_n$$

$$\Rightarrow \quad T_n = 5+\frac{(n-1)\times[2\times 6+(n-2)\times 2]}{2}$$

$$\Rightarrow \quad T_n = 5+(n-1)(n+4)$$

$$\Rightarrow \quad T_n = n^2+3n+1.$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^2+3k+1) = \sum_{k=1}^{n} k^2+3\sum_{k=1}^{n} k+n$$

$$= \frac{1}{6}n(n+1)(2n+1)+3\times\frac{1}{2}n(n+1)+n$$

$$= \frac{1}{3}n(n+2)(n+4).$$

EXAMPLE 12 *Find the sum of the first n terms of the series* $3+7+13+21+31+\ldots$.

SOLUTION Let $S_n = 3+7+13+21+31+\ldots+T_{n-1}+T_n$

$S_n = \quad 3+\ \ 7+13+21+\ldots+T_{n-2}+T_{n-1}+T_n$

On subtraction, we get

$$0 = 3+[4+6+8+10+\ldots \text{ to } (n-1) \text{ terms}]-T_n$$

$$\Rightarrow \quad T_n = 3+[4+6+8+\ldots \text{ to } (n-1) \text{ terms}]$$

$$= 3+\frac{(n-1)}{2}\times[2\times 4+(n-2)\times 2]$$

$$= 3+(n-1)(n+2) = (n^2+n+1).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^2+k+1)$$

$$= \sum_{k=1}^{n} k^2+\sum_{k=1}^{n} k+n \qquad [\because\ 1+1+1+\ldots \text{ to } n \text{ terms} = n]$$

$$= \frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)+n = \frac{1}{3}n(n^2+3n+5).$$

EXAMPLE 13 *Find the 50th term of the series* $2+3+6+11+18+\ldots$.

SOLUTION Let $S_n = 2+3+6+11+18+\ldots+T_{n-1}+T_n$.

Again $S_n = \quad 2+3+\ \ 6+11+\ldots+T_{n-1}+T_n$.

On subtracting, we get

$$0 = 2+[1+3+5+7+\ldots \text{ to } (n-1) \text{ terms}]-T_n$$

$$\Rightarrow \quad T_n = 2+[1+3+5+7+\ldots \text{ to } (n-1) \text{ terms}]$$

$$= 2+\frac{(n-1)}{2}\cdot[2\times 1+(n-2)\times 2] = (n-1)^2+2.$$

$$\Rightarrow \quad T_{50} = \{(50-1)^2+2\} = \{(49)^2+2\} = (2401+2) = 2403.$$

EXAMPLE 14 *Find the sum of the series*

$$(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\ldots \text{ to (i) } n \text{ terms} \quad \text{(ii) } 10 \text{ terms.}$$

SOLUTION We have

$$T_k=\{(2k+1)^3-(2k)^3\}$$
$$=\{8k^3+1+6k(2k+1)-8k^3\}=(12k^2+6k+1).$$

(i) sum to n terms is given by

$$S_n=12\sum_{k=1}^{n}k^2+6\sum_{k=1}^{n}k+n$$
$$=12\times\frac{1}{6}n(n+1)(2n+1)+6\times\frac{1}{2}n(n+1)+n$$
$$=2n(n+1)(2n+1)+3n(n+1)+n$$
$$=n(n+1)[4n+2+3]+n=n[(n+1)(4n+5)+1]$$
$$=n(4n^2+9n+6)=(4n^3+9n^2+6n).$$

(ii) Sum to 10 terms is given by

$$S_{10}=10\times[4\times10^2+9\times10+6]=4960.$$

EXERCISE 13A

Find the sum of the series whose nth term is given by:

1. $(3n^2+2n)$ **2.** $n(n+1)(n+4)$

3. $(4n^3+6n^2+2n)$ **4.** $(3n^2-3n+2)$

5. $(2n^2-3n+5)$ **6.** (n^3-3^n)

Find the sum of the series:

7. $(2^2+4^2+6^2+8^2+\ldots$ to n terms$)$

8. $(2^3+4^3+6^3+8^3+\ldots$ to n terms$)$

9. $(5^2+6^2+7^2+\ldots+20^2)$

10. $1\times2+2\times3+3\times4+4\times5+\ldots$ to n terms

11. $3\times8+6\times11+9\times14+\ldots$ to n terms

12. $1\times2^2+2\times3^2+3\times4^2+\ldots$ to n terms

13. $1\times2^2+3\times3^2+5\times4^2+\ldots$ to n terms

14. $3\times1^2+5\times2^2+7\times3^2+\ldots$ to n terms

15. $(1\times2\times3)+(2\times3\times4)+(3\times4\times5)+\ldots$ to n terms

16. $(1\times2\times4)+(2\times3\times7)+(3\times4\times10)+\ldots$ to n terms

17. $\frac{1}{(1\times2)}+\frac{1}{(2\times3)}+\frac{1}{(3\times4)}+\ldots$ to n terms

18. $\frac{1}{(1\times3)}+\frac{1}{(3\times5)}+\frac{1}{(5\times7)}+\ldots+\frac{1}{(2n-1)(2n+1)}$

19. $\frac{1}{(1\times6)}+\frac{1}{(6\times11)}+\frac{1}{(11\times16)}+\ldots+\frac{1}{(5n-4)(5n+1)}$

20. $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$ to n terms

21. $3+15+35+63+\ldots$ to n terms

22. $1+5+12+22+35+\ldots$ to n terms

23. $5+7+13+31+85+\ldots$ to n terms

24. If $S_k=\frac{(1+2+3+\ldots+k)}{k}$, prove that

$$(S_1^2+S_2^2+S_3^2+\ldots+S_n^2)=\frac{1}{24}n(2n^2+9n+13).$$

25. If S_n denotes the sum of the cubes of the first n natural numbers and s_n denotes the sum of the first n natural numbers then find the value of $\sum_{k=1}^{n}\frac{S_k}{s_k}$.

ANSWERS (EXERCISE 13A)

1. $\frac{1}{2}n(n+1)(2n+3)$ **2.** $\frac{1}{12}n(n+1)(n+2)(3n+17)$ **3.** $n(n+1)^2(n+2)$

4. $2n(n^2+1)$ **5.** $\frac{n}{6}(4n^2-3n+23)$ **6.** $\frac{1}{4}n^2(n+1)^2-\frac{3}{2}(3^n-1)$

7. $\frac{2}{3}n(n+1)(2n+1)$ **8.** $2n^2(n+1)^2$ **9.** 2840

10. $\frac{1}{3}n(n+1)(n+2)$ **11.** $3n(n+1)(n+3)$ **12.** $\frac{n(n+1)(n+2)(3n+5)}{12}$

13. $\frac{n}{2}(n^3+4n^2+4n-1)$ **14.** $\frac{n(n+1)(3n^2+5n+1)}{6}$ **15.** $\frac{n(n+1)(n+2)(n+3)}{4}$

16. $\frac{n(n+1)(3n^2+19n+14)}{12}$ **17.** $\frac{n}{(n+1)}$ **18.** $\frac{n}{(2n+1)}$ **19.** $\frac{n}{(5n+1)}$

20. $\frac{1}{24}n(2n^2+9n+13)$ **21.** $\frac{1}{3}n(4n^2+6n-1)$ **22.** $\frac{1}{2}n^2(n+1)$

23. $\frac{1}{2}(3^n+8n-1)$ **25.** $\frac{1}{6}n(n+1)(n+2)$

HINTS TO SOME SELECTED QUESTIONS

4. $T_n=(3n^2-3n+2)$.

$\therefore \quad S_n=3\left(\sum_{k=1}^{n}k^2\right)-3\left(\sum_{k=1}^{n}k\right)+2n$ [$\because$ $2+2+2+\ldots n$ times $=2n$].

6. $T_n=(n^3-3^n)$.

$\therefore \quad S_n=\left(\sum_{k=1}^{n}k^3\right)-\left(\sum_{k=1}^{n}3^k\right)=\frac{1}{4}n^2(n+1)^2-\{3+3^2+\ldots+3^n\}$

$\Rightarrow \quad S_n=\left\{\frac{1}{4}n^2(n+1)^2-\frac{3(3^n-1)}{(3-1)}\right\}$.

7. $T_n=\{2+(n-1)\times 2\}^2=(2n)^2=4n^2$.

$\therefore \quad S_n=\sum_{k=1}^{n}T_k=4\left(\sum_{k=1}^{n}k^2\right)=4\times\frac{1}{6}n(n+1)(2n+1)$.

8. $T_n = \{2 + (n-1) \times 2\}^3 = (2n)^3 = 8n^3.$

$\therefore \quad S_n = 8\left\{\sum_{k=1}^{n} k^3\right\} = 8 \times \frac{1}{4} n^2 (n+1)^2 = 2n^2 (n+1)^2.$

9. Given series $= (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \ldots + 20^2) - (1^2 + 2^2 + 3^2 + 4^2)$

$= \left\{\frac{1}{6} n(n+1)(2n+1), \text{ when } n = 20\right\} - 30$

$= \left(\frac{1}{6} \times 20 \times 21 \times 41\right) - 30 = 2840.$

10. $T_n = n(n+1) = n^2 + n.$

$\therefore \quad S_n = \left(\sum_{k=1}^{n} k^2\right) + \left(\sum_{k=1}^{n} k\right).$

11. $T_n = (n\text{th term of } 3, 6, 9, \ldots) \times (n\text{th term of } 8, 11, 14, \ldots)$

$= \{3 + (n-1) \times 3\} \times \{8 + (n-1) \times 3\} = 3n(3n+5) = 9n^2 + 15n.$

$\therefore \quad S_n = 9\left(\sum_{k=1}^{n} k^2\right) + 15\left(\sum_{k=1}^{n} k\right).$

12. $T_n = (n\text{th term of } 1, 2, 3, \ldots) \times (n\text{th term of } 2^2, 3^2, 4^2, \ldots)$

$= n \times (n+1)^2 = n \times (n^2 + 2n + 1) = n^3 + 2n^2 + n.$

$\therefore \quad S_n = \left(\sum_{k=1}^{n} k^3\right) + 2\left(\sum_{k=1}^{n} k^2\right) + \left(\sum_{k=1}^{n} k\right).$

13. $T_n = (n\text{th term of } 1, 3, 5, \ldots) \times (n\text{th term of } 2^2, 3^2, 4^2, \ldots)$

$= \{1 + (n-1) \times 2\} \times (n+1)^2 = (2n-1)(n^2 + 2n + 1) = (2n^3 + 3n^2 - 1).$

$\therefore \quad S_n = 2\left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) - n.$

14. $T_n = (n\text{th term of } 3, 5, 7, \ldots) \times (n\text{th term of } 1^2, 2^2, 3^2, \ldots)$

$= \{3 + (n-1) \times 2\} \times n^2 = (2n+1) n^2 = (2n^3 + n^2).$

$\therefore \quad S_n = 2\left(\sum_{k=1}^{n} k^3\right) + \left(\sum_{k=1}^{n} k^2\right).$

15. $T_n = n(n+1)(n+2) = (n^3 + 3n^2 + 2n).$

$\therefore \quad S_n = \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right).$

16. $T_n = (n\text{th term of } 1, 2, 3, \ldots) \times (n\text{th term of } 2, 3, 4, \ldots) \times (n\text{th term of } 4, 7, 10, \ldots)$

$= n(n+1) \times [4 + (n-1) \times 3] = n(n+1)(3n+1) = (3n^3 + 4n^2 + n).$

17. $T_n = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1) + Bn}{n(n+1)} = \frac{(A+B)n + A}{n(n+1)}.$

$\therefore \quad A + B = 0 \text{ and } A = 1 \Rightarrow A = 1 \text{ and } B = -1$

$\therefore \quad T_n = \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$\therefore \quad T_1 = \left(\frac{1}{1} - \frac{1}{2}\right)$

$T_2 = \left(\frac{1}{2} - \frac{1}{3}\right)$

$T_3 = \left(\frac{1}{3} - \frac{1}{4}\right)$

… … … …

… … … …

$$T_{n-1} = \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$T_n = \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Adding columnwise, we get

$$S_n = (T_1 + T_2 + T_3 + \ldots + T_n) = \left(1 - \frac{1}{n+1}\right) = \frac{(n+1)-1}{n+1} = \frac{n}{n+1}.$$

18. $T_n = \dfrac{1}{(2n-1)(2n+1)} = \dfrac{A}{2n-1} + \dfrac{B}{2n+1} = \dfrac{A(2n+1) + B(2n-1)}{(2n-1)(2n+1)} = \dfrac{2(A+B)n + (A-B)}{(2n-1)(2n+1)}.$

$\therefore \quad 2(A+B) = 0$ and $A - B = 1 \Rightarrow A + B = 0, A - B = 1 \Rightarrow A = \dfrac{1}{2}, B = \dfrac{-1}{2}.$

$\therefore \quad T_n = \dfrac{1}{2} \times \left\{\dfrac{1}{2n-1} - \dfrac{1}{2n+1}\right\}.$

$\therefore \quad T_1 = \dfrac{1}{2} \times \left(\dfrac{1}{1} - \dfrac{1}{3}\right)$

$$T_2 = \frac{1}{2} \times \left(\frac{1}{3} - \frac{1}{5}\right)$$

$$T_3 = \frac{1}{2} \times \left(\frac{1}{5} - \frac{1}{7}\right)$$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$$T_{n-1} = \frac{1}{2} \times \left\{\frac{1}{2n-3} - \frac{1}{2n-1}\right\}$$

$$T_n = \frac{1}{2} \times \left\{\frac{1}{2n-1} - \frac{1}{2n+1}\right\}.$$

Adding columnwise, we get

$$S_n = (T_1 + T_2 + T_3 + \ldots + T_n) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{2n+1-1}{2(2n+1)} = \frac{n}{(2n+1)}.$$

19. $T_n = \dfrac{1}{5} \times \left\{\dfrac{1}{5n-4} - \dfrac{1}{5n+1}\right\}.$

$\therefore \quad T_1 = \dfrac{1}{5} \times \left\{\dfrac{1}{1} - \dfrac{1}{6}\right\}$

$$T_2 = \frac{1}{5} \times \left\{\frac{1}{6} - \frac{1}{11}\right\}$$

$$T_3 = \frac{1}{5} \times \left\{\frac{1}{11} - \frac{1}{16}\right\}$$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$$T_{n-1} = \frac{1}{5} \times \left\{\frac{1}{5n-9} - \frac{1}{5n-4}\right\}$$

$$T_n = \frac{1}{5} \times \left\{\frac{1}{5n-4} - \frac{1}{5n+1}\right\}.$$

Adding columnwise, we get

$$S_n = (T_1 + T_2 + \ldots + T_n) = \frac{1}{5}\left(1 - \frac{1}{5n+1}\right) = \frac{(5n+1-1)}{5(5n+1)} = \frac{n}{(5n+1)}.$$

20. $T_n = \dfrac{1^3 + 2^3 + 3^3 + \ldots + n^3}{1 + 3 + 5 + \ldots + (2n-1)} = \dfrac{\left\{\frac{1}{2}n(n+1)\right\}^2}{\frac{n}{2}\{1 + (2n-1)\}} = \dfrac{(n+1)^2}{4} = \dfrac{1}{4}n^2 + \dfrac{1}{2}n + \dfrac{1}{4}.$

$\therefore \quad S_n = \sum_{k=1}^{n} T_k$

$$= \frac{1}{4}\left(\sum_{k=1}^{n} k^2\right) + \frac{1}{2}\left(\sum_{k=1}^{n} k\right) + \frac{1}{4}n.$$

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21. $S_n = 3 + 15 + 35 + 63 + \ldots + T_{n-1} + T_n$ … (i)

$S_n = \quad 3 + 15 + 35 + \ldots + T_{n-2} + T_{n-1} + T_n.$ … (ii)

On subtracting (ii) from (i), we get

$$0 = 3 + \{12 + 20 + 28 + \ldots \text{ to } (n-1) \text{ terms}\} - T_n$$

$$\Rightarrow \quad T_n = 3 + \frac{1}{2}(n-1)\{2 \times 12 + (n-2) \times 8\} = (4n^2 - 1)$$

$$\Rightarrow \quad S_n = 4\left(\sum_{k=1}^{n} k^2\right) - n = \left\{4 \times \frac{1}{6} n(n+1)(2n+1)\right\} - n = \frac{1}{3} n(4n^2 + 6n - 1).$$

24. $S_k = \dfrac{(1 + 2 + 3 + \ldots + k)}{k} = \dfrac{k(k+1)}{2k} = \dfrac{k+1}{2}.$

$$\therefore \quad \sum_{k=1}^{n} S_k^2 = \sum_{k=1}^{n} \frac{1}{4}(k+1)^2 = \frac{1}{4}\left(\sum_{k=1}^{n} k^2\right) + \frac{1}{2}\left(\sum_{k=1}^{n} k\right) + \frac{1}{4}(1 + 1 + \ldots n \text{ times})$$

$$= \frac{1}{4} \times \frac{1}{6} n(n+1)(2n+1) + \frac{1}{4} n(n+1) + \frac{1}{4} n$$

$$= \frac{n}{24}(2n^2 + 9n + 13) \qquad \text{[on simplification].}$$

25. $S_k = \left\{\sum_{r=1}^{k} r^3\right\} = \left\{\frac{1}{2}k(k+1)\right\}^2$ and $s_k = \left\{\sum_{r=1}^{k} r\right\} = \frac{1}{2}k(k+1).$

$$\therefore \quad \frac{S_k}{s_k} = \frac{\left\{\frac{1}{2}k(k+1)\right\}^2}{\frac{1}{2}k(k+1)} = \frac{1}{2}k(k+1) = \frac{1}{2}k^2 + \frac{1}{2}k$$

$$\Rightarrow \quad \sum_{k=1}^{n} \frac{S_k}{s_k} = \frac{1}{2}\left\{\sum_{k=1}^{n} k^2\right\} + \frac{1}{2}\left\{\sum_{k=1}^{n} k\right\}$$

$$= \frac{1}{2} \times \frac{1}{6} \times n(n+1)(2n+1) + \frac{1}{2} \times \frac{1}{2} n(n+1)$$

$$= \frac{1}{12}\{n(n+1)(2n+1) + 3n(n+1)\} = \frac{1}{6} n(n+1)(n+2).$$

EXERCISE 13B

Very-Short-Answer Questions

1. Find the sum $(2 + 4 + 6 + 8 + \ldots + 100)$.
2. Find the sum $(41 + 42 + 43 + \ldots + 100)$.
3. Find the sum $\{(11)^2 + (12)^2 + (13)^2 + \ldots + (20)^2\}$.
4. Find the sum $\{(6)^3 + (7)^3 + (8)^3 + (9)^3 + (10)^3\}$.
5. If $\sum_{k=1}^{n} k = 210$, find the value of $\sum_{k=1}^{n} k^2$.
6. If $\sum_{k=1}^{n} k = 45$, find the value of $\sum_{k=1}^{n} k^3$.
7. Find the sum of the series $\{2^2 + 4^2 + 6^2 + \ldots + (2n)^2\}$.
8. Find the sum of 10 terms of the geomeric series $\sqrt{2} + \sqrt{6} + \sqrt{18} + \ldots$.
9. Find the sum of n terms of the series whose rth term is $(r + 2^r)$.

ANSWERS (EXERCISE 13B)

1. 2550 **2.** 4230 **3.** 2485 **4.** 2800 **5.** 2870 **6.** 2025

7. $\frac{2}{3}n(n+1)(2n+1)$ **8.** $121(\sqrt{6}+\sqrt{2})$ **9.** $\frac{1}{2}n(n+1)+2^{n+1}-2$

HINTS TO SOME SELECTED QUESTIONS

1. We know that $(1+2+3+\ldots+n)=\frac{1}{2}n(n+1)$.

$\therefore \quad (2+4+6+8+\ldots+100)=2\times(1+2+3+\ldots+50)=\left(2\times\frac{1}{2}\times 50\times 51\right)=2550.$

2. Given sum $=(1+2+3+\ldots+40+41+\ldots+100)-(1+2+3+\ldots+40)$

$=\left(\frac{1}{2}\times 100\times 101\right)-\left(\frac{1}{2}\times 40\times 41\right)=(5050-820)=4230.$

3. We know that $(1^2+2^2+3^2+\ldots+n^2)=\frac{1}{6}n(n+1)(2n+1)$.

$\therefore \quad$ given sum $=\{1^2+2^2+\ldots+(10)^2+(11)^2+\ldots+(20)^2\}-\{1^2+2^2+\ldots+(10)^2\}$

$=\left(\frac{1}{6}\times 20\times 21\times 41\right)-\left(\frac{1}{6}\times 10\times 11\times 21\right)=(2870-385)=2485.$

4. We know that $(1^3+2^3+3^3+\ldots+n^3)=\left\{\frac{1}{2}n(n+1)\right\}^2$.

$\therefore \quad$ given sum $=\{1^3+2^3+3^3+4^3+5^3+6^3+\ldots+(10)^3\}-\{1^3+2^3+3^3+4^3+5^3\}$

$=\left(\frac{1}{2}\times 10\times 11\right)^2-\left(\frac{1}{2}\times 5\times 6\right)^2=(55)^2-(15)^2=(3025-225)=2800.$

5. $(1+2+3+\ldots+n)=210 \Rightarrow \frac{1}{2}n(n+1)=210 \Rightarrow n^2+n-420=0$

$\Rightarrow (n+21)(n-20)=0 \Rightarrow n=20.$

$\therefore \quad (1^2+2^2+3^2+\ldots+n^2)=\frac{1}{6}n(n+1)(2n+1)$, where $n=20$

$=\left(\frac{1}{6}\times 20\times 21\times 41\right)=2870.$

6. Given $(1+2+3+\ldots+n)=45 \Rightarrow \frac{1}{2}n(n+1)=45.$

$\therefore \quad \left(\sum_{k=1}^{n}k^3\right)=\left\{\frac{1}{2}n(n+1)\right\}^2=(45)^2=2025.$

7. $T_k=(2k)^2=4k^2.$

$\therefore \quad S_n=\sum_{k=1}^{n}T_k=4\left(\sum_{k=1}^{n}k^2\right)=\left\{4\times\frac{1}{6}n(n+1)(2n+1)\right\}=\frac{2}{3}n(n+1)(2n+1).$

8. Given series is a geometric series in which $a=\sqrt{2}$, $r=\sqrt{3}$ and $n=10$.

$\therefore \quad S=\frac{a(r^n-1)}{(r-1)}=\frac{\sqrt{2}\{(\sqrt{3})^{10}-1\}}{(\sqrt{3}-1)}\times\frac{(\sqrt{3}+1)}{(\sqrt{3}+1)}=\frac{\sqrt{2}(243-1)(\sqrt{3}+1)}{2}$

$\Rightarrow \quad S=121(\sqrt{6}+\sqrt{2}).$

9. Required sum $=\sum_{r=1}^{n}T_r=\sum_{r=1}^{n}(r+2^r)$

$=\sum_{r=1}^{n}r+\sum_{r=1}^{n}2^r=\frac{1}{2}n(n+1)+(2+2^2+2^3+\ldots+2^n)$

$=\frac{1}{2}n(n+1)+\frac{2(2^n-1)}{(2-1)}=\frac{1}{2}n(n+1)+2^{n+1}-2.$

13
Some Special Series

SUM OF FIRST *n* NATURAL NUMBERS

THEOREM 1 *Prove that* $(1+2+3+\ldots+n)=\frac{1}{2}n(n+1)$

i.e., $\sum_{k=1}^{n} k=\frac{1}{2}n(n+1)$.

PROOF Consider the series $(1+2+3+\ldots+n)$.

This is an arithmetic series in which $a=1$, $d=1$ and $l=n$.

$\therefore \quad S_n=\frac{n}{2}(1+n) \Rightarrow S_n=\frac{1}{2}n(n+1)$ $\left[\because S_n=\frac{1}{2}(a+l)\right]$.

Hence, $\sum_{k=1}^{n} k=\frac{1}{2}n(n+1)$.

SUM OF THE SQUARES OF FIRST *n* NATURAL NUMBERS

THEOREM 2 *Prove that* $(1^2+2^2+3^2+\ldots+n^2)=\frac{1}{6}n(n+1)(2n+1)$

i.e., $\sum_{k=1}^{n} k^2=\frac{1}{6}n(n+1)(2n+1)$.

PROOF Let $S_n=1^2+2^2+3^2+\ldots+n^2$.

Consider the identity: $k^3-(k-1)^3=3k^2-3k+1$. ... (i)

Putting $k=1, 2, 3, \ldots, (n-1), n$ successively in (i), we get

$1^3-0^3=3\cdot 1^2-3\cdot 1+1;$

$2^3-1^3=3\cdot 2^2-3\cdot 2+1;$

$3^3-2^3=3\cdot 3^2-3\cdot 3+1;$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$(n-1)^3-(n-2)^3=3\cdot(n-1)^2-3\cdot(n-1)+1;$

$n^3-(n-1)^3=3\cdot n^2-3\cdot n+1.$

Adding columnwise, we get

$(n^3-0^3)=3(1^2+2^2+3^2+\ldots+n^2)-3(1+2+3+\ldots+n)+n$

$\Rightarrow \quad n^3=3\cdot\sum_{k=1}^{n} k^2-3\cdot\frac{1}{2}n(n+1)+n \quad \left[\because (1+2+3+\ldots+n)=\frac{1}{2}n(n+1)\right]$

$\Rightarrow \quad 3\left(\sum_{k=1}^{n} k^2\right)=n^3+\frac{3}{2}n(n+1)-n=\frac{1}{2}(2n^3+3n^2+n)$

$$\Rightarrow \quad \left[\sum_{k=1}^{n} k^2\right] = \frac{1}{6}(2n^3+3n^2+n) = \frac{1}{6}n(2n^2+3n+1) = \frac{1}{6}n(n+1)(2n+1).$$

Hence, $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

SUM OF THE CUBES OF FIRST *n* NATURAL NUMBERS

THEOREM 3 *Prove that* $(1^3+2^3+3^3+\ldots+n^3) = \left\{\frac{1}{2}n(n+1)\right\}^2$

i.e., $\sum_{k=1}^{n} k^3 = \left\{\frac{1}{2}n(n+1)\right\}^2$.

PROOF Let $S_n = 1^3+2^3+3^3+\ldots+n^3$.

Consider the identity: $k^4-(k-1)^4 = 4k^3-6k^2+4k-1$. … (i)

Putting $k = 1, 2, 3, \ldots, (n-1), n$ successively in (i), we get

$$1^4-0^4 = 4\cdot 1^3-6\cdot 1^2+4\cdot 1-1$$
$$2^4-1^4 = 4\cdot 2^3-6\cdot 2^2+4\cdot 2-1$$
$$3^4-2^4 = 4\cdot 3^3-6\cdot 3^2+4\cdot 3-1$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$(n-1)^4-(n-2)^4 = 4\cdot(n-1)^3-6\cdot(n-1)^2+4\cdot(n-1)-1$$
$$n^4-(n-1)^4 = 4\cdot n^3-6\cdot n^2+4\cdot n-1.$$

Adding columnwise, we get

$$(n^4-0^4) = 4\cdot\{1^3+2^3+3^3+\ldots+n^3\}-6\cdot\{1^2+2^2+3^2+\ldots+n^2)$$
$$+4\cdot\{1+2+3+\ldots+n\}-(1+1+1+\ldots \text{to } n \text{ terms})$$

$$\Rightarrow \quad n^4 = 4\cdot\left(\sum_{k=1}^{n} k^3\right)-6\cdot\left(\sum_{k=1}^{n} k^2\right)+4\cdot\left(\sum_{k=1}^{n} k\right)-n$$

$$\Rightarrow \quad n^4 = 4\cdot\left(\sum_{k=1}^{n} k^3\right)-6\cdot\frac{1}{6}n(n+1)(2n+1)+4\cdot\frac{1}{2}n(n+1)-n$$

$$\left\{\because \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \text{ and } \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)\right\}$$

$$\Rightarrow \quad 4\cdot\left(\sum_{k=1}^{n} k^3\right) = n^4+n(n+1)(2n+1)-2n(n+1)+n$$

$$\Rightarrow \quad \left(\sum_{k=1}^{n} k^3\right) = \frac{1}{4}\cdot(n^4+2n^3+n^2) = \frac{1}{4}n^2(n^2+2n+1) = \left\{\frac{1}{2}n(n+1)\right\}^2$$

$$\therefore \quad \left(\sum_{k=1}^{n} k^3\right) = \left\{\frac{1}{2}n(n+1)\right\}^2$$

i.e., $(1^3+2^3+3^3+\ldots+n^3) = \left\{\frac{1}{2}n(n+1)\right\}^2$.

SUM OF THE CUBES OF FIRST *n* ODD NATURAL NUMBERS

THEOREM 4 *Prove that* $[1^3+3^3+5^3+\ldots(2n-1)^3] = n^2(2n^2-1)$

i.e., $\sum_{k=1}^{n} (2k-1)^3 = n^2(2n^2-1)$.

PROOF Let $S_n = 1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$.

Then, $S_n = \sum_{k=1}^{n} (2k-1)^3$

$= \sum_{k=1}^{n} (8k^3 - 12k^2 + 6k - 1)$

$= 8\left(\sum_{k=1}^{n} k^3\right) - 12\left(\sum_{k=1}^{n} k^2\right) + 6\left(\sum_{k=1}^{n} k\right) - n$ [$\because$ $1 + 1 + \ldots$ to n terms $= n$]

$= 8 \cdot \frac{1}{4} n^2 (n+1)^2 - 12 \cdot \frac{1}{6} n(n+1)(2n+1) + 6 \cdot \frac{1}{2} n(n+1) - n$

$= 2n^2 (n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$

$= n(n+1) \cdot \{2n(n+1) - 2(2n+1) + 3\} - n$

$= n(n+1)(2n^2 - 2n + 1) - n = n \cdot \{(n+1)(2n^2 - 2n + 1) - 1\}$

$= n(2n^3 - n) = n^2 (2n^2 - 1)$.

$\therefore \quad \{1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3\} = n^2 (2n^2 - 1)$.

SUMMARY

(i) $(1 + 2 + 3 + \ldots + n) = \frac{1}{2} n(n+1)$

i.e., $\left(\sum_{k=1}^{n} k\right) = \frac{1}{2} n(n+1)$.

(ii) $(1^2 + 2^2 + 3^2 + \ldots + n^2) = \frac{1}{6} n(n+1)(2n+1)$

i.e., $\left(\sum_{k=1}^{n} k^2\right) = \frac{1}{6} n(n+1)(2n+1)$.

(iii) $(1^3 + 2^3 + 3^3 + \ldots + n^3) = \left\{\frac{1}{2} n(n+1)\right\}^2$

i.e., $\left(\sum_{k=1}^{n} k^3\right) = \left\{\frac{1}{2} n(n+1)\right\}^2$.

(iv) $\{1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3\} = n^2 (2n^2 - 1)$

i.e., $\sum_{k=1}^{n} (2k-1)^3 = n^2 (2n^2 - 1)$.

SOLVED EXAMLES

EXAMPLE 1 *If S_1, S_2 and S_3 are the sums of first n natural numbers, their squares and their cubes respectively then show that $9S_2^2 = S_3(1 + 8S_1)$.*

SOLUTION We have

$S_1 = (1 + 2 + 3 + \ldots + n) \Rightarrow S_1 = \frac{1}{2} n(n+1)$;

$S_2 = (1^2 + 2^2 + 3^2 + \ldots + n^2) \Rightarrow S_2 = \frac{1}{6} n(n+1)(2n+1)$;

and $S_3 = (1^2 + 2^3 + 3^3 + \ldots + n^3) \Rightarrow S_3 = \frac{1}{4} n^2 (n+1)^2$.

$$\therefore \quad 9S_2^2 = 9 \times \frac{1}{36} \cdot n^2(n+1)^2(2n+1)^2 = \frac{1}{4}n^2(n+1)^2(2n+1)^2.$$

And, $S_3(1+8S_1) = \frac{1}{4}n^2(n+1)^2 \cdot \left\{1 + 8 \cdot \frac{1}{2}n(n+1)\right\}$

$$= \frac{1}{4}n^2(n+1)^2(4n^2+4n+1) = \frac{1}{4}n^2(n+1)^2(2n+1)^2.$$

Hence, $9S_2^2 = S_3(1+8S_1)$.

EXAMPLE 2 *Find the sum to n terms of the series whose nth term is* $n(n+3)$.

SOLUTION We have, $T_k = k(k+3) = (k^2+3k)$.

$\therefore$ sum to *n* terms is given by

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n}(k^2+3k) = \sum_{k=1}^{n}k^2 + 3\sum_{k=1}^{n}k$$

$$= \frac{1}{6}n(n+1)(2n+1) + 3 \cdot \frac{1}{2}n(n+1) = \frac{1}{6}\{n(n+1)(2n+1) + 9n(n+1)\}$$

$$= \frac{1}{6}n(n+1)(2n+1+9) = \frac{1}{6}n(n+1) \cdot 2(n+5) = \frac{1}{3}n(n+1)(n+5).$$

Hence, the required sum is $\frac{1}{3}n(n+1)(n+5)$.

EXAMPLE 3 *Find the sum to n terms of the series whose nth term is* $(2n-1)^2$.

SOLUTION We have, $T_k = (2k-1)^2 = (4k^2-4k+1)$.

$\therefore$ sum to *n* terms is given by

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n}(4k^2-4k+1) = 4 \cdot \sum_{k=1}^{n}k^2 - 4 \cdot \sum_{k=1}^{n}k + n$$

$$[\because\ 1+1+\ldots n \text{ times} = n]$$

$$= 4 \cdot \frac{1}{6}n(n+1)(2n+1) - 4 \cdot \frac{1}{2}n(n+1) + n$$

$$= \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n$$

$$= \frac{1}{3} \cdot \{2n(n+1)(2n+1) - 6n(n+1) + 3n\}$$

$$= \frac{1}{3}[n(n+1)\{2(2n+1) - 6\} + 3n] = \frac{1}{3} \cdot [4n(n+1)(n-1) + 3n]$$

$$= \frac{1}{3}(4n^3 - n) = \frac{1}{3}n(4n^2-1).$$

Hence, the required sum is $\frac{1}{3}n(4n^2-1)$.

EXAMPLE 4 *Find the sum to n terms of the series whose nth term is* (n^2+2^n).

SOLUTION We have, $T_k = (k^2+2^k)$.

$\therefore$ $S_n = \sum_{k=1}^{n} T_k$

$$= \sum_{k=1}^{n}(k^2+2^k) = \sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}2^k$$

$$= \frac{1}{6}n(n+1)(2n+1) + (2 + 2^2 + 2^3 + \ldots + 2^n)$$

$$\left[\because \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)\right]$$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{2(2^n - 1)}{(2-1)} \quad [\because\ 2 + 2^2 + \ldots + 2^n \text{ is a GP}]$$

$$= \frac{1}{6}n(n+1)(2n+1) + 2(2^n - 1).$$

Hence, the required sum is $\frac{1}{6}n(n+1)(2n+1) + 2(2^n - 1)$.

EXAMPLE 5 *Sum the series* $3 \cdot 8 + 6 \cdot 11 + 9 \cdot 14 + \ldots$ *to n terms.*

SOLUTION We have

$$T_k = (k\text{th term of } 3, 6, 9, \ldots) \times (k\text{th term of } 8, 11, 14, \ldots)$$

$$= \{3 + (k-1) \times 3\} \times \{8 + (k-1) \times 3\} = 3k(3k+5)$$

$$= (9k^2 + 15k).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (9k^2 + 15k) = 9\left(\sum_{k=1}^{n} k^2\right) + 15\left(\sum_{k=1}^{n} k\right)$$

$$= 9 \cdot \left\{\frac{1}{6}n(n+1)(2n+1)\right\} + 15 \cdot \left\{\frac{1}{2}n(n+1)\right\}$$

$$= \frac{3}{2}n(n+1)\{(2n+1) + 5\} = 3n(n+1)(n+3).$$

Hence, the required sum is $3n(n+1)(n+3)$.

EXAMPLE 6 *Sum the series* $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots$ *to n terms.*

SOLUTION We have

$$T_k = (k\text{th term of } 1, 2, 3, \ldots) \times (k\text{th term of } 2, 3, 4, \ldots)$$

$$\times (k\text{th term of } 3, 4, 5, \ldots)$$

$$= \{1 + (k-1) \times 1\} + \{2 + (k-1) \times 1\} \times \{3 + (k-1) \times 1\}$$

$$= k(k+1)(k+2) = (k^3 + 3k^2 + 2k).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (k^3 + 3k^2 + 2k) = \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k$$

$$= \frac{1}{4}n^2(n+1)^2 + 3 \cdot \frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1)$$

$$\left\{\because \sum_{k=1}^{n} k^3 = \left\{\frac{1}{2}n(n+1)\right\}^2, \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1), \right.$$

$$\left. \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)\right\}$$

$$= \frac{1}{4}n(n+1)\{n(n+1) + 2(2n+1) + 4\}$$

$$= \frac{1}{4}n(n+1)(n^2+5n+6) = \frac{1}{4}n(n+1)(n+2)(n+3).$$

Hence, the required sum is $\frac{1}{4}n(n+1)(n+2)(n+3)$.

EXAMPLE 7 *Find the sum of the series* $1^2+3^2+5^2+\ldots$ *to n terms.*

SOLUTION We have

$$T_k = k\text{th term of } (1^2+3^2+5^2+\ldots)$$
$$= \{1+(k-1)\times 2\}^2 = (2k-1)^2 = (4k^2-4k+1).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n}(4k^2-4k+1)$$
$$= 4\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + (1+1+\ldots n \text{ times})$$
$$= 4\cdot\frac{1}{6}n(n+1)(2n+1) - 4\cdot\frac{1}{2}n(n+1) + n$$
$$= \frac{n}{3}\cdot\{2(n+1)(2n+1) - 6(n+1) + 3\} = \frac{n}{3}(4n^2-1).$$

Hence, the required sum is $\frac{n}{3}(4n^2-1)$.

EXAMPLE 8 *Find the sum* $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots$ *to n terms.*

SOLUTION We have

$$T_k = (1^2+2^2+3^2+\ldots+k^2) = \left\{\frac{1}{6}k(k+1)(2k+1)\right\} = \frac{1}{6}(2k^3+3k^2+k).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$
$$= \frac{1}{6}\cdot\sum_{k=1}^{n}(2k^3+3k^2+k) = \frac{1}{6}\cdot 2\cdot\sum_{k=1}^{n}k^3 + \frac{1}{6}\cdot 3\cdot\sum_{k=1}^{n}k^2 + \frac{1}{6}\cdot\sum_{k=1}^{n}k$$
$$= \frac{1}{3}\cdot\sum_{k=1}^{n}k^3 + \frac{1}{2}\cdot\sum_{k=1}^{n}k^2 + \frac{1}{6}\cdot\sum_{k=1}^{n}k$$
$$= \frac{1}{3}\cdot\left\{\frac{1}{2}n(n+1)\right\}^2 + \frac{1}{2}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{1}{6}\cdot\frac{1}{2}n(n+1)$$
$$= \frac{1}{12}n^2(n+1)^2 + \frac{1}{12}n(n+1)(2n+1) + \frac{1}{12}n(n+1)$$
$$= \frac{1}{12}n(n+1)\cdot\{n(n+1)+(2n+1)+1\} = \frac{1}{12}n(n+1)^2(n+2).$$

Hence, the required sum is $\frac{1}{12}n(n+1)^2(n+2)$.

EXAMPLE 9 *Prove that* $\dfrac{1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)} = \dfrac{3n+5}{3n+1}$.

SOLUTION We have

kth term of the numerator $= k(k+1)^2 = (k^3+2k^2+k)$.

kth term of the denominator $= k^2(k+1) = (k^3+k^2)$.

$$\therefore \quad \text{LHS} = \frac{\sum_{k=1}^{n}(k^3+2k^2+k)}{\sum_{k=1}^{n}(k^3+k^2)} = \frac{\left(\sum_{k=1}^{n}k^3\right)+2\cdot\left(\sum_{k=1}^{n}k^2\right)+\left(\sum_{k=1}^{n}k\right)}{\left(\sum_{k=1}^{n}k^3\right)+\left(\sum_{k=1}^{n}k^2\right)}$$

$$= \frac{\frac{1}{4}n^2(n+1)^2+2\cdot\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)}{\frac{1}{4}n^2(n+1)^2+\frac{1}{6}n(n+1)(2n+1)}$$

$$\left\{\because \left(\sum_{k=1}^{n}k\right)=\frac{1}{2}n(n+1);\left(\sum_{k=1}^{n}k^2\right)=\frac{1}{6}n(n+1)(2n+1)\right.$$

$$\left.\text{and}\left(\sum_{k=1}^{n}k^3\right)=\frac{1}{4}n^2(n+1)^2\right\}$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12}\times\frac{12}{n(n+1)(n+2)(3n+1)}$$

[on simplifying]

$$= \frac{(3n+5)}{(3n+1)} = \text{RHS.}$$

EXAMPLE 10 *Find the sum of n terms of the series*

$$\frac{1}{(2\times5)}+\frac{1}{(5\times8)}+\frac{1}{(8\times11)}+\ldots.$$

SOLUTION We have

$$T_k = \frac{1}{(k\text{th term of } 2, 5, 8, \ldots)\times(k\text{th term of } 5, 8, 11, \ldots)}$$

$$= \frac{1}{\{2+(k-1)\times3\}\times\{5+(k-1)\times3\}}$$

$$= \frac{1}{(3k-1)(3k+2)} = \frac{1}{3}\left\{\frac{1}{(3k-1)}-\frac{1}{(3k+2)}\right\}.$$

$$\therefore \quad T_k = \frac{1}{3}\left\{\frac{1}{(3k-1)}-\frac{1}{(3k+2)}\right\}. \qquad \ldots \text{(i)}$$

Putting $k = 1, 2, 3, \ldots, n$ successively in (i), we get

$$T_1 = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)$$

$$T_2 = \frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)$$

$$T_3 = \frac{1}{3}\left(\frac{1}{8}-\frac{1}{11}\right)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$T_n = \frac{1}{3}\left\{\frac{1}{(3n-1)}-\frac{1}{(3n+2)}\right\}.$$

Adding columnwise, we get

$$S_n = (T_1+T_2+T_3+\ldots+T_n)$$

$$= \frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right) = \frac{n}{2(3n+2)}.$$

EXAMPLE 11 *Find the sum of n terms of the series* $5+11+19+29+41+\ldots$.

SOLUTION Let $S_n = 5+11+19+29+\ldots+T_{n-1}+T_n$

$S_n = \quad 5+11+19+\ldots+T_{n-2}+T_{n-1}+T_n$

On subtraction, we get

$$0 = 5+[6+8+10+12+\ldots \text{ to } (n-1) \text{ terms}]-T_n$$

$$\Rightarrow \quad T_n = 5+\frac{(n-1)\times[2\times 6+(n-2)\times 2]}{2}$$

$$\Rightarrow \quad T_n = 5+(n-1)(n+4)$$

$$\Rightarrow \quad T_n = n^2+3n+1.$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^2+3k+1) = \sum_{k=1}^{n} k^2+3\sum_{k=1}^{n} k+n$$

$$= \frac{1}{6}n(n+1)(2n+1)+3\times\frac{1}{2}n(n+1)+n$$

$$= \frac{1}{3}n(n+2)(n+4).$$

EXAMPLE 12 *Find the sum of the first n terms of the series* $3+7+13+21+31+\ldots$.

SOLUTION Let $S_n = 3+7+13+21+31+\ldots+T_{n-1}+T_n$

$S_n = \quad 3+\ \ 7+13+21+\ldots+T_{n-2}+T_{n-1}+T_n$

On subtraction, we get

$$0 = 3+[4+6+8+10+\ldots \text{ to } (n-1) \text{ terms}]-T_n$$

$$\Rightarrow \quad T_n = 3+[4+6+8+\ldots \text{ to } (n-1) \text{ terms}]$$

$$= 3+\frac{(n-1)}{2}\times[2\times 4+(n-2)\times 2]$$

$$= 3+(n-1)(n+2) = (n^2+n+1).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^2+k+1)$$

$$= \sum_{k=1}^{n} k^2+\sum_{k=1}^{n} k+n \qquad [\because\ 1+1+1+\ldots \text{ to } n \text{ terms} = n]$$

$$= \frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)+n = \frac{1}{3}n(n^2+3n+5).$$

EXAMPLE 13 *Find the 50th term of the series* $2+3+6+11+18+\ldots$.

SOLUTION Let $S_n = 2+3+6+11+18+\ldots+T_{n-1}+T_n$.

Again $S_n = \quad 2+3+\ \ 6+11+\ldots+T_{n-1}+T_n$.

On subtracting, we get

$$0 = 2+[1+3+5+7+\ldots \text{ to } (n-1) \text{ terms}]-T_n$$

$$\Rightarrow \quad T_n = 2+[1+3+5+7+\ldots \text{ to } (n-1) \text{ terms}]$$

$$= 2+\frac{(n-1)}{2}\cdot[2\times 1+(n-2)\times 2] = (n-1)^2+2.$$

$$\Rightarrow \quad T_{50} = \{(50-1)^2+2\} = \{(49)^2+2\} = (2401+2) = 2403.$$

EXAMPLE 14 *Find the sum of the series*

$$(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\ldots \text{ to (i) } n \text{ terms} \quad \text{(ii) } 10 \text{ terms.}$$

SOLUTION We have

$$T_k=\{(2k+1)^3-(2k)^3\}$$
$$=\{8k^3+1+6k(2k+1)-8k^3\}=(12k^2+6k+1).$$

(i) sum to n terms is given by

$$S_n=12\sum_{k=1}^{n}k^2+6\sum_{k=1}^{n}k+n$$
$$=12\times\frac{1}{6}n(n+1)(2n+1)+6\times\frac{1}{2}n(n+1)+n$$
$$=2n(n+1)(2n+1)+3n(n+1)+n$$
$$=n(n+1)[4n+2+3]+n=n[(n+1)(4n+5)+1]$$
$$=n(4n^2+9n+6)=(4n^3+9n^2+6n).$$

(ii) Sum to 10 terms is given by

$$S_{10}=10\times[4\times10^2+9\times10+6]=4960.$$

EXERCISE 13A

Find the sum of the series whose nth term is given by:

1. $(3n^2+2n)$ **2.** $n(n+1)(n+4)$

3. $(4n^3+6n^2+2n)$ **4.** $(3n^2-3n+2)$

5. $(2n^2-3n+5)$ **6.** (n^3-3^n)

Find the sum of the series:

7. $(2^2+4^2+6^2+8^2+\ldots$ to n terms$)$

8. $(2^3+4^3+6^3+8^3+\ldots$ to n terms$)$

9. $(5^2+6^2+7^2+\ldots+20^2)$

10. $1\times2+2\times3+3\times4+4\times5+\ldots$ to n terms

11. $3\times8+6\times11+9\times14+\ldots$ to n terms

12. $1\times2^2+2\times3^2+3\times4^2+\ldots$ to n terms

13. $1\times2^2+3\times3^2+5\times4^2+\ldots$ to n terms

14. $3\times1^2+5\times2^2+7\times3^2+\ldots$ to n terms

15. $(1\times2\times3)+(2\times3\times4)+(3\times4\times5)+\ldots$ to n terms

16. $(1\times2\times4)+(2\times3\times7)+(3\times4\times10)+\ldots$ to n terms

17. $\frac{1}{(1\times2)}+\frac{1}{(2\times3)}+\frac{1}{(3\times4)}+\ldots$ to n terms

18. $\frac{1}{(1\times3)}+\frac{1}{(3\times5)}+\frac{1}{(5\times7)}+\ldots+\frac{1}{(2n-1)(2n+1)}$

19. $\frac{1}{(1\times6)}+\frac{1}{(6\times11)}+\frac{1}{(11\times16)}+\ldots+\frac{1}{(5n-4)(5n+1)}$

20. $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$ to n terms

21. $3+15+35+63+\ldots$ to n terms

22. $1+5+12+22+35+\ldots$ to n terms

23. $5+7+13+31+85+\ldots$ to n terms

24. If $S_k=\frac{(1+2+3+\ldots+k)}{k}$, prove that

$$(S_1^2+S_2^2+S_3^2+\ldots+S_n^2)=\frac{1}{24}n(2n^2+9n+13).$$

25. If S_n denotes the sum of the cubes of the first n natural numbers and s_n denotes the sum of the first n natural numbers then find the value of $\sum_{k=1}^{n}\frac{S_k}{s_k}$.

ANSWERS (EXERCISE 13A)

1. $\frac{1}{2}n(n+1)(2n+3)$ **2.** $\frac{1}{12}n(n+1)(n+2)(3n+17)$ **3.** $n(n+1)^2(n+2)$

4. $2n(n^2+1)$ **5.** $\frac{n}{6}(4n^2-3n+23)$ **6.** $\frac{1}{4}n^2(n+1)^2-\frac{3}{2}(3^n-1)$

7. $\frac{2}{3}n(n+1)(2n+1)$ **8.** $2n^2(n+1)^2$ **9.** 2840

10. $\frac{1}{3}n(n+1)(n+2)$ **11.** $3n(n+1)(n+3)$ **12.** $\frac{n(n+1)(n+2)(3n+5)}{12}$

13. $\frac{n}{2}(n^3+4n^2+4n-1)$ **14.** $\frac{n(n+1)(3n^2+5n+1)}{6}$ **15.** $\frac{n(n+1)(n+2)(n+3)}{4}$

16. $\frac{n(n+1)(3n^2+19n+14)}{12}$ **17.** $\frac{n}{(n+1)}$ **18.** $\frac{n}{(2n+1)}$ **19.** $\frac{n}{(5n+1)}$

20. $\frac{1}{24}n(2n^2+9n+13)$ **21.** $\frac{1}{3}n(4n^2+6n-1)$ **22.** $\frac{1}{2}n^2(n+1)$

23. $\frac{1}{2}(3^n+8n-1)$ **25.** $\frac{1}{6}n(n+1)(n+2)$

HINTS TO SOME SELECTED QUESTIONS

4. $T_n=(3n^2-3n+2)$.

$\therefore \quad S_n=3\left(\sum_{k=1}^{n}k^2\right)-3\left(\sum_{k=1}^{n}k\right)+2n$ [$\because$ $2+2+2+\ldots n$ times $=2n$].

6. $T_n=(n^3-3^n)$.

$\therefore \quad S_n=\left(\sum_{k=1}^{n}k^3\right)-\left(\sum_{k=1}^{n}3^k\right)=\frac{1}{4}n^2(n+1)^2-\{3+3^2+\ldots+3^n\}$

$\Rightarrow \quad S_n=\left\{\frac{1}{4}n^2(n+1)^2-\frac{3(3^n-1)}{(3-1)}\right\}$.

7. $T_n=\{2+(n-1)\times 2\}^2=(2n)^2=4n^2$.

$\therefore \quad S_n=\sum_{k=1}^{n}T_k=4\left(\sum_{k=1}^{n}k^2\right)=4\times\frac{1}{6}n(n+1)(2n+1)$.

8. $T_n = \{2 + (n-1) \times 2\}^3 = (2n)^3 = 8n^3.$

$\therefore \quad S_n = 8\left\{\sum_{k=1}^{n} k^3\right\} = 8 \times \frac{1}{4} n^2 (n+1)^2 = 2n^2 (n+1)^2.$

9. Given series $= (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \ldots + 20^2) - (1^2 + 2^2 + 3^2 + 4^2)$

$= \left\{\frac{1}{6} n(n+1)(2n+1), \text{ when } n = 20\right\} - 30$

$= \left(\frac{1}{6} \times 20 \times 21 \times 41\right) - 30 = 2840.$

10. $T_n = n(n+1) = n^2 + n.$

$\therefore \quad S_n = \left(\sum_{k=1}^{n} k^2\right) + \left(\sum_{k=1}^{n} k\right).$

11. $T_n = (n\text{th term of } 3, 6, 9, \ldots) \times (n\text{th term of } 8, 11, 14, \ldots)$

$= \{3 + (n-1) \times 3\} \times \{8 + (n-1) \times 3\} = 3n(3n+5) = 9n^2 + 15n.$

$\therefore \quad S_n = 9\left(\sum_{k=1}^{n} k^2\right) + 15\left(\sum_{k=1}^{n} k\right).$

12. $T_n = (n\text{th term of } 1, 2, 3, \ldots) \times (n\text{th term of } 2^2, 3^2, 4^2, \ldots)$

$= n \times (n+1)^2 = n \times (n^2 + 2n + 1) = n^3 + 2n^2 + n.$

$\therefore \quad S_n = \left(\sum_{k=1}^{n} k^3\right) + 2\left(\sum_{k=1}^{n} k^2\right) + \left(\sum_{k=1}^{n} k\right).$

13. $T_n = (n\text{th term of } 1, 3, 5, \ldots) \times (n\text{th term of } 2^2, 3^2, 4^2, \ldots)$

$= \{1 + (n-1) \times 2\} \times (n+1)^2 = (2n-1)(n^2 + 2n + 1) = (2n^3 + 3n^2 - 1).$

$\therefore \quad S_n = 2\left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) - n.$

14. $T_n = (n\text{th term of } 3, 5, 7, \ldots) \times (n\text{th term of } 1^2, 2^2, 3^2, \ldots)$

$= \{3 + (n-1) \times 2\} \times n^2 = (2n+1) n^2 = (2n^3 + n^2).$

$\therefore \quad S_n = 2\left(\sum_{k=1}^{n} k^3\right) + \left(\sum_{k=1}^{n} k^2\right).$

15. $T_n = n(n+1)(n+2) = (n^3 + 3n^2 + 2n).$

$\therefore \quad S_n = \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right).$

16. $T_n = (n\text{th term of } 1, 2, 3, \ldots) \times (n\text{th term of } 2, 3, 4, \ldots) \times (n\text{th term of } 4, 7, 10, \ldots)$

$= n(n+1) \times [4 + (n-1) \times 3] = n(n+1)(3n+1) = (3n^3 + 4n^2 + n).$

17. $T_n = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1) + Bn}{n(n+1)} = \frac{(A+B)n + A}{n(n+1)}.$

$\therefore \quad A + B = 0 \text{ and } A = 1 \Rightarrow A = 1 \text{ and } B = -1$

$\therefore \quad T_n = \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$\therefore \quad T_1 = \left(\frac{1}{1} - \frac{1}{2}\right)$

$T_2 = \left(\frac{1}{2} - \frac{1}{3}\right)$

$T_3 = \left(\frac{1}{3} - \frac{1}{4}\right)$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$$T_{n-1} = \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$T_n = \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Adding columnwise, we get

$$S_n = (T_1 + T_2 + T_3 + \ldots + T_n) = \left(1 - \frac{1}{n+1}\right) = \frac{(n+1)-1}{n+1} = \frac{n}{n+1}.$$

18. $T_n = \dfrac{1}{(2n-1)(2n+1)} = \dfrac{A}{2n-1} + \dfrac{B}{2n+1} = \dfrac{A(2n+1) + B(2n-1)}{(2n-1)(2n+1)} = \dfrac{2(A+B)n + (A-B)}{(2n-1)(2n+1)}.$

$\therefore \quad 2(A+B) = 0$ and $A - B = 1 \Rightarrow A + B = 0, A - B = 1 \Rightarrow A = \dfrac{1}{2}, B = \dfrac{-1}{2}.$

$\therefore \quad T_n = \dfrac{1}{2} \times \left\{\dfrac{1}{2n-1} - \dfrac{1}{2n+1}\right\}.$

$\therefore \quad T_1 = \dfrac{1}{2} \times \left(\dfrac{1}{1} - \dfrac{1}{3}\right)$

$$T_2 = \frac{1}{2} \times \left(\frac{1}{3} - \frac{1}{5}\right)$$

$$T_3 = \frac{1}{2} \times \left(\frac{1}{5} - \frac{1}{7}\right)$$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$$T_{n-1} = \frac{1}{2} \times \left\{\frac{1}{2n-3} - \frac{1}{2n-1}\right\}$$

$$T_n = \frac{1}{2} \times \left\{\frac{1}{2n-1} - \frac{1}{2n+1}\right\}.$$

Adding columnwise, we get

$$S_n = (T_1 + T_2 + T_3 + \ldots + T_n) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{2n+1-1}{2(2n+1)} = \frac{n}{(2n+1)}.$$

19. $T_n = \dfrac{1}{5} \times \left\{\dfrac{1}{5n-4} - \dfrac{1}{5n+1}\right\}.$

$\therefore \quad T_1 = \dfrac{1}{5} \times \left\{\dfrac{1}{1} - \dfrac{1}{6}\right\}$

$$T_2 = \frac{1}{5} \times \left\{\frac{1}{6} - \frac{1}{11}\right\}$$

$$T_3 = \frac{1}{5} \times \left\{\frac{1}{11} - \frac{1}{16}\right\}$$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$$T_{n-1} = \frac{1}{5} \times \left\{\frac{1}{5n-9} - \frac{1}{5n-4}\right\}$$

$$T_n = \frac{1}{5} \times \left\{\frac{1}{5n-4} - \frac{1}{5n+1}\right\}.$$

Adding columnwise, we get

$$S_n = (T_1 + T_2 + \ldots + T_n) = \frac{1}{5}\left(1 - \frac{1}{5n+1}\right) = \frac{(5n+1-1)}{5(5n+1)} = \frac{n}{(5n+1)}.$$

20. $T_n = \dfrac{1^3 + 2^3 + 3^3 + \ldots + n^3}{1 + 3 + 5 + \ldots + (2n-1)} = \dfrac{\left\{\frac{1}{2}n(n+1)\right\}^2}{\frac{n}{2}\{1 + (2n-1)\}} = \dfrac{(n+1)^2}{4} = \dfrac{1}{4}n^2 + \dfrac{1}{2}n + \dfrac{1}{4}.$

$\therefore \quad S_n = \sum_{k=1}^{n} T_k$

$$= \frac{1}{4}\left(\sum_{k=1}^{n} k^2\right) + \frac{1}{2}\left(\sum_{k=1}^{n} k\right) + \frac{1}{4}n.$$

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21. $S_n = 3 + 15 + 35 + 63 + \ldots + T_{n-1} + T_n$ … (i)

$S_n = \quad 3 + 15 + 35 + \ldots + T_{n-2} + T_{n-1} + T_n.$ … (ii)

On subtracting (ii) from (i), we get

$$0 = 3 + \{12 + 20 + 28 + \ldots \text{ to } (n-1) \text{ terms}\} - T_n$$

$$\Rightarrow \quad T_n = 3 + \frac{1}{2}(n-1)\{2 \times 12 + (n-2) \times 8\} = (4n^2 - 1)$$

$$\Rightarrow \quad S_n = 4\left(\sum_{k=1}^{n} k^2\right) - n = \left\{4 \times \frac{1}{6} n(n+1)(2n+1)\right\} - n = \frac{1}{3} n(4n^2 + 6n - 1).$$

24. $S_k = \dfrac{(1 + 2 + 3 + \ldots + k)}{k} = \dfrac{k(k+1)}{2k} = \dfrac{k+1}{2}.$

$$\therefore \quad \sum_{k=1}^{n} S_k^2 = \sum_{k=1}^{n} \frac{1}{4}(k+1)^2 = \frac{1}{4}\left(\sum_{k=1}^{n} k^2\right) + \frac{1}{2}\left(\sum_{k=1}^{n} k\right) + \frac{1}{4}(1 + 1 + \ldots n \text{ times})$$

$$= \frac{1}{4} \times \frac{1}{6} n(n+1)(2n+1) + \frac{1}{4} n(n+1) + \frac{1}{4} n$$

$$= \frac{n}{24}(2n^2 + 9n + 13) \qquad \text{[on simplification].}$$

25. $S_k = \left\{\sum_{r=1}^{k} r^3\right\} = \left\{\frac{1}{2}k(k+1)\right\}^2$ and $s_k = \left\{\sum_{r=1}^{k} r\right\} = \frac{1}{2}k(k+1).$

$$\therefore \quad \frac{S_k}{s_k} = \frac{\left\{\frac{1}{2}k(k+1)\right\}^2}{\frac{1}{2}k(k+1)} = \frac{1}{2}k(k+1) = \frac{1}{2}k^2 + \frac{1}{2}k$$

$$\Rightarrow \quad \sum_{k=1}^{n} \frac{S_k}{s_k} = \frac{1}{2}\left\{\sum_{k=1}^{n} k^2\right\} + \frac{1}{2}\left\{\sum_{k=1}^{n} k\right\}$$

$$= \frac{1}{2} \times \frac{1}{6} \times n(n+1)(2n+1) + \frac{1}{2} \times \frac{1}{2} n(n+1)$$

$$= \frac{1}{12}\{n(n+1)(2n+1) + 3n(n+1)\} = \frac{1}{6} n(n+1)(n+2).$$

EXERCISE 13B

Very-Short-Answer Questions

1. Find the sum $(2 + 4 + 6 + 8 + \ldots + 100)$.
2. Find the sum $(41 + 42 + 43 + \ldots + 100)$.
3. Find the sum $\{(11)^2 + (12)^2 + (13)^2 + \ldots + (20)^2\}$.
4. Find the sum $\{(6)^3 + (7)^3 + (8)^3 + (9)^3 + (10)^3\}$.
5. If $\sum_{k=1}^{n} k = 210$, find the value of $\sum_{k=1}^{n} k^2$.
6. If $\sum_{k=1}^{n} k = 45$, find the value of $\sum_{k=1}^{n} k^3$.
7. Find the sum of the series $\{2^2 + 4^2 + 6^2 + \ldots + (2n)^2\}$.
8. Find the sum of 10 terms of the geomeric series $\sqrt{2} + \sqrt{6} + \sqrt{18} + \ldots$.
9. Find the sum of n terms of the series whose rth term is $(r + 2^r)$.

ANSWERS (EXERCISE 13B)

1. 2550 **2.** 4230 **3.** 2485 **4.** 2800 **5.** 2870 **6.** 2025

7. $\frac{2}{3}n(n+1)(2n+1)$ **8.** $121(\sqrt{6}+\sqrt{2})$ **9.** $\frac{1}{2}n(n+1)+2^{n+1}-2$

HINTS TO SOME SELECTED QUESTIONS

1. We know that $(1+2+3+\ldots+n)=\frac{1}{2}n(n+1)$.

$\therefore \quad (2+4+6+8+\ldots+100)=2\times(1+2+3+\ldots+50)=\left(2\times\frac{1}{2}\times 50\times 51\right)=2550.$

2. Given sum $=(1+2+3+\ldots+40+41+\ldots+100)-(1+2+3+\ldots+40)$

$=\left(\frac{1}{2}\times 100\times 101\right)-\left(\frac{1}{2}\times 40\times 41\right)=(5050-820)=4230.$

3. We know that $(1^2+2^2+3^2+\ldots+n^2)=\frac{1}{6}n(n+1)(2n+1)$.

$\therefore \quad$ given sum $=\{1^2+2^2+\ldots+(10)^2+(11)^2+\ldots+(20)^2\}-\{1^2+2^2+\ldots+(10)^2\}$

$=\left(\frac{1}{6}\times 20\times 21\times 41\right)-\left(\frac{1}{6}\times 10\times 11\times 21\right)=(2870-385)=2485.$

4. We know that $(1^3+2^3+3^3+\ldots+n^3)=\left\{\frac{1}{2}n(n+1)\right\}^2$.

$\therefore \quad$ given sum $=\{1^3+2^3+3^3+4^3+5^3+6^3+\ldots+(10)^3\}-\{1^3+2^3+3^3+4^3+5^3\}$

$=\left(\frac{1}{2}\times 10\times 11\right)^2-\left(\frac{1}{2}\times 5\times 6\right)^2=(55)^2-(15)^2=(3025-225)=2800.$

5. $(1+2+3+\ldots+n)=210 \Rightarrow \frac{1}{2}n(n+1)=210 \Rightarrow n^2+n-420=0$

$\Rightarrow (n+21)(n-20)=0 \Rightarrow n=20.$

$\therefore \quad (1^2+2^2+3^2+\ldots+n^2)=\frac{1}{6}n(n+1)(2n+1)$, where $n=20$

$=\left(\frac{1}{6}\times 20\times 21\times 41\right)=2870.$

6. Given $(1+2+3+\ldots+n)=45 \Rightarrow \frac{1}{2}n(n+1)=45.$

$\therefore \quad \left(\sum_{k=1}^{n}k^3\right)=\left\{\frac{1}{2}n(n+1)\right\}^2=(45)^2=2025.$

7. $T_k=(2k)^2=4k^2.$

$\therefore \quad S_n=\sum_{k=1}^{n}T_k=4\left(\sum_{k=1}^{n}k^2\right)=\left\{4\times\frac{1}{6}n(n+1)(2n+1)\right\}=\frac{2}{3}n(n+1)(2n+1).$

8. Given series is a geometric series in which $a=\sqrt{2}$, $r=\sqrt{3}$ and $n=10$.

$\therefore \quad S=\frac{a(r^n-1)}{(r-1)}=\frac{\sqrt{2}\{(\sqrt{3})^{10}-1\}}{(\sqrt{3}-1)}\times\frac{(\sqrt{3}+1)}{(\sqrt{3}+1)}=\frac{\sqrt{2}(243-1)(\sqrt{3}+1)}{2}$

$\Rightarrow \quad S=121(\sqrt{6}+\sqrt{2}).$

9. Required sum $=\sum_{r=1}^{n}T_r=\sum_{r=1}^{n}(r+2^r)$

$=\sum_{r=1}^{n}r+\sum_{r=1}^{n}2^r=\frac{1}{2}n(n+1)+(2+2^2+2^3+\ldots+2^n)$

$=\frac{1}{2}n(n+1)+\frac{2(2^n-1)}{(2-1)}=\frac{1}{2}n(n+1)+2^{n+1}-2.$